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ANALYTIC EQUIVALENCE OF ALGEBROID CURVES

ANDREW H. WALLACE

1. Introduction. Let k be an algebraically closed field and let $x_1, x_2, \ldots, x_n$ be indeterminates. Denote by R_n the ring $k[[x_1, x_2, \ldots, x_n]]$ of power series in the x_i with coefficients in the field k. Let $\mathfrak{A}$ and $\mathfrak{A}'$ be two ideals in this ring. Then $\mathfrak{A}$ and $\mathfrak{A}'$ will be said to be analytically equivalent if there is an automorphism T of R_n such that $T(\mathfrak{A}) = \mathfrak{A}'$. $\mathfrak{A}$ and $\mathfrak{A}'$ will be called analytically equivalent under T.

The above situation can be described geometrically as follows. The ideals $\mathfrak{A}$ and $\mathfrak{A}'$ can be regarded as defining algebroid varieties V and V' in $(x_1, x_2, \ldots, x_n)$-space, and these varieties will be said to be analytically equivalent under T.

The automorphism T can be expressed by means of equations of the form:

$$T(x_i) = \sum a_{ij}x_j + f_i(x)$$

where the determinant $|a_{ij}|$ is not zero and the f_i are power series of order not less than two (that is to say, containing terms of degree two or more only). If the f_i are all of order greater than or equal to r, the analytic equivalence T will be said to be of order r. Throughout this paper the only analytic equivalences which will be considered will be those in which the coefficients a_{ij} in the above equations satisfy the conditions $a_{ii} = 1$, $a_{ij} = 0$ for $i \neq j$. This will not, in fact, impose any essential restriction, for in the main theorem to be proved the analytic equivalence which appears happens in any case to be of this form.

The problem to be studied here can be formulated as follows. Suppose that $F_i(x) = 0$, $i = 1, 2, \ldots, r$ and $F_i'(x) = 0$, $i = 1, 2, \ldots, r$ are sets of equations for the varieties V and V' respectively, that is to say that F_i and the F_i' are sets of generators of the ideals $\mathfrak{A}$ and $\mathfrak{A}'$ respectively. Then if V and V' are analytically equivalent under an analytic equivalence of sufficiently high order, it is clear that the F_i' can be chosen as power series differing from the corresponding F_i only by terms of high order. The question is, to what extent is the converse of this statement true?

A partial answer to this is given by a theorem of Samuel **(4)** for the case in which $r = 1$ and the origin is an isolated singular point of the variety V. Under these conditions Samuel's theorem states that, if the order of $F_1 - F_1'$ is high enough, then the hypersurfaces V and V' are analytically equivalent. The restrictions on the singularities of V here are very strong, and are imposed by the method of proof. On the other hand it is clear that if V and V' are to be

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analytically equivalent some restrictions on the relations between their singular points must be imposed. However it seems to be rather difficult to see exactly what these conditions should be for varieties of arbitrary dimension; and so a general answer to the question formulated above seems impossible until some new methods are found.

In view of the difficulties mentioned I am restricting myself here to the attempt to answer the above question in the case of curves. The form which the answer takes in this case makes it sufficient to consider curves V and V' defined by the sets of equations $F_i = 0$ and $F_i' = 0$, respectively, i running in each case from 1 to $n - 1$. The question to be answered then becomes the following. V and V' being as just described, will they be analytically equivalent if the orders of the $F_i - F_i'$ are high enough? Samuel's theorem gives the answer yes when $n = 2$ and V is irreducible. It is clear however that in general the answer will not be affirmative without some further restriction. For if V has a multiple component, geometrically speaking a component singular on one of the $F_i = 0$ or along which two of them touch, it is intuitively obvious that modification of the equations of V, even by terms of high order, may cause such a component to split, thus making analytic equivalence impossible. But it seems reasonable to hope that components of V along which the F_i intersect simply and transversally will be carried by an analytic equivalence into similar components of V' if the orders of the $F_i - F_i'$ are high enough. This is the main result to be proved in this paper. The components of V just described correspond to the isolated prime components of the ideal $\mathfrak{A}$. The corresponding algebraic formulation of the result indicated will appear below as Theorem 2.

Since it is not impossible that the problem treated here may sometime receive an answer in the case of varieties of dimension greater than 1, some of the auxiliary results are treated with greater generality than is actually needed for the present paper, in the hope that they may be useful for a more general treatment.

The proof of the main theorem will be carried out by induction on the dimension of the ambient space, the step from $n - 1$ to n being made by means of a suitable projection. Changing the notation, denote the ideals $\mathfrak{A}$ and $\mathfrak{A}'$ by (F) and (F′), with generators F_i and F_i' respectively, $i = 1, 2, \ldots, r$. The first step is to show that the F_i and F_i' can be taken as polynomials in x_n. Having done this let H_i be the resultant, with respect to x_n, of F_i and F_r and let (H) be the ideal generated in $R_{n-1} = k[[x_1, x_2, \ldots, x_{n-1}]]$ by the H_i. Define (H') similarly by means of the F_i'. It will then be shown that, if the co-ordinates have been suitably chosen, and after a suitable adjustment of the F_i, the isolated prime components of (F) project into isolated prime components of (H). Here the projection of an ideal in R_n means its intersection with R_{n-1}. When $r = n - 1$, the induction hypothesis will then imply that the intersection (G) of the isolated primes of (H) can be carried by an analytic equivalence in R_{n-1} into the intersection (G') of certain components of (H'),

provided that the orders of the $F_i - F_i'$, and so (as will be shown) of the $H_j - H_j'$, are sufficiently high. The next step is to prove that, again if the orders of the $F_i - F_i'$ are high enough, this analytic equivalence can be extended to one in R_n carrying (G, F_r) into (G', F_r'). Now the isolated primes of (F) will be components of (G, F_r) and so will be carried by the extended analytic equivalence into components of (G', F_r'). It remains to be shown that these will in fact be components of (F') if the orders of the $F_i - F_i'$ are high enough. The proof of this will be based on the fact that, in terms of a suitable topology, an analytic equivalence affects the components of (F) continuously, and that the only components of (G', F_r') which are then sufficiently near to those of (F) are already components of (F').

2. Preliminary adjustments. One of the main objects of this section is to show that the series F_i can be assumed to be polynomials in x_n. This is justified by means of the Weierstrass Preparation Theorem, which can be stated as follows:

Let F be an element of R_n and let it be of order m, and suppose a linear change of co-ordinates has, if necessary, been made so that F contains the term x_n^m with a non-zero coefficient. Then there is a power series P in R_n of order 0 *(that is to say a unit of R_n) such that PF is a polynomial $x_n^m + a_1x_n^{m-1} + \ldots + a_m$ in x_n with coefficients in R_{n-1}. Also, since PF is of order m, a_i is of order not less than i.*

The classical case of this theorem applies, of course, to the case where k is the field of complex numbers, but the proof can be given entirely in terms of formal power series over any field (**1**, p. 183 ff.). If this formal algebraic proof is examined, it will be observed that the terms of various degrees of the series P are determined step by step, and that the terms up to any given degree depend only on the terms of F up to a certain degree. It follows that a complement to the above theorem can be stated, namely:

If F is as above and F' is a second series such that $F - F'$ is of sufficiently high order, then the series P and P' of order zero can be found such that $PF = x_n^m + a_1x_n^{m-1} + \ldots + a_m$ and $P'F' = x_n^m + a_1'x_n^{m-1} + \ldots + a_m'$ where the orders of the $a_i - a_i'$ are greater than a preassigned integer.

Now let (F) be the ideal generated as in the introduction by $F_1, F_2, \ldots, F_r$. Co-ordinates are to be chosen, if necessary after a linear change of variables, so that the following conditions hold:

(1) The Weierstrass Preparation Theorem can be applied to all the F_i, which can therefore be assumed to be replaced by polynomials, without changing the ideal (F).

(2) If $\mathfrak{p}$ is an isolated $(n - r)$-dimensional prime component of (F) then none of the series $\partial F_i/\partial x_n$ $(i = 1, 2, \ldots, r)$, is in $\mathfrak{p}$.

(3) No two $(n-r)$-dimensional components of (F) have the same projection in R_{n-1}. This is equivalent to the geometrical statement that no two $(n-r)$-dimensional components of the algebroid variety V have the same projection on $x_n = 0$.

It will now be checked that the co-ordinates can be chosen so that these conditions are satisfied. For these verifications it should be assumed that k has an infinite number of elements, or, if not, that the linear changes of variables made have generic coefficients, that is to say, independent indeterminates over k.

To make condition (1) hold it is sufficient to change the variables so that each F_i contains among its terms of lowest degree a power of x_n with a non-zero coefficient.

To show that condition (2) can be made to hold, let $\mathfrak{o}$ be the quotient ring of R_n with respect to $\mathfrak{p}$, that is to say, the neighbourhood ring of the algebroid variety whose ideal is $\mathfrak{p}$. Then $\mathfrak{o}$ is a regular local ring of dimension r (**2**, p. 33) with maximal ideal $\mathfrak{o}\mathfrak{p}$. This ideal is generated by the r elements F_i, which therefore form a system of parameters. It follows (**2**, p. 34) that the Jacobian matrix $(\partial F_i/\partial x_j)$ is of rank r mod $\mathfrak{p}$. In particular, for any given i, all the $\partial F_i/\partial x_k$ cannot be zero. Making the change of variables $x_i = \sum a_{ij}\bar{x}_j$, it follows that $\partial F_i/\partial \bar{x}_n = \sum \partial F_i/\partial x_k a_{kn}$ is not zero for suitably chosen a_{kn}. Thus for suitably chosen a_{ij}, $\partial F_i/\partial \bar{x}_n \neq 0$ mod $\mathfrak{p}$ for each i. Assume the a_{ij} chosen in this way, make the appropriate co-ordinate change, and drop the bars over the new co-ordinates.

The verification of condition (3) is equivalent to proving a special case of the following theorem: *If a number of varieties are given in n-space, then co-ordinates can be chosen so that no two of them have the same projections on* $x_n = 0$. To indicate the method of proof it will be sufficient to consider the case of two distinct varieties V_1 and V_2 in n-space. Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be the corresponding ideals in R_n. Let $\mathfrak{A}_2'$ be the ideal in $k[[y_1, y_2, \ldots, y_n]]$ obtained from $\mathfrak{A}_2$ by substituting the y_i for the corresponding x_i. Then the ideal $(\mathfrak{A}_1, \mathfrak{A}_2')$ in $k[[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n]]$ defines the product variety $V_1 \times V_2$ in $2n$-space. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be independent indeterminates and write $\phi_i = (x_n - y_n)/\lambda_n - (x_i - y_i)/\lambda_i$ for $i = 1, 2, \ldots, n-1$. Then the ideal $(\mathfrak{A}_1, \mathfrak{A}_2', \phi_1, \ldots, \phi_{n-1})$ defines a subvariety of $V_1 \times V_2$ whose points are pairs (p_1, p_2) with $p_i \in V_i$ such that p_1 and p_2 project on the same point of $x_n = 0$ along the "direction" $(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Calculating the Jacobian matrix of a set of generators of $(\mathfrak{A}_1, \mathfrak{A}_2', \phi_1, \ldots, \phi_{n-1})$ and using Proposition 2, (**2**, p. 34), it follows easily that the dimension of each component of this subvariety is less than the common dimension of V_1 and V_2 (it is, of course, sufficient to consider the case where the dimensions of V_1 and V_2 are the same). A suitable change of co-ordinates then gives the required result.

The three conditions above have been considered separately, but it is easy to see that they can be made to hold simultaneously.

The co-ordinates having been chosen as above, it is now necessary to make

certain adjustments to the generators of the ideal (F). Let the u_{ij}, $(i, j = 1$ $2, \ldots, r)$, be independent indeterminates and write

$$\hat{F}_i = \sum_{j=1}^{r} u_{ij}F_j, \qquad i = 1, 2, \ldots, r-1,$$
$$\hat{F}_r = F_r$$

and let $\hat{k}$ be the algebraic closure of $k(u)$, the field obtained from k by adjoining the u_{ij}. Let H_i be the resultant of $\hat{F}_i$ and $\hat{F}_r$ with respect to x_n, and let (H) denote the ideal in $\hat{k}[[x_1, x_2, \ldots, x_{n-1}]]$ generated by the $H_i (i = 1, 2, \ldots, r-1)$.

A component of (H) will be said to be independent of (u) if it has generators in $k[[x_1, x_2, \ldots, x_n]]$. For example, it is not hard to see that a component of the ideal in $\hat{k}[[x_1, x_2, \ldots, x_n]]$ generated by the $\hat{F}_i$, and so by the F_i, will project into a component of (H) independent of (u). What is important for the present purpose is essentially the converse of this result. Namely:

LEMMA 1. *If $\mathfrak{p}$ is a prime component of (H) independent of (u), then, for any i, a common root of the equations $\hat{F}_i = 0$ and $\hat{F}_r = 0$, both regarded as polynomial equations in x_n with the coefficients reduced mod $\mathfrak{p}$, is necessarily a common root of all the $\hat{F}_j = 0$ with coefficients reduced mod $\mathfrak{p}$.*

Proof. For it can be assumed that the u_{ij} are independent indeterminates over the field of fractions of $k[[x_1, x_2, \ldots, x_{n-1}]]/\mathfrak{p}$. Any root of $\hat{F}_r = F_r = 0$ reduced mod $\mathfrak{p}$ is algebraic over this field, and so, if it is a root of $\hat{F}_i = 0$, reduced mod $\mathfrak{p}$, it must be a root of all the equations $F_j = 0$, reduced mod $\mathfrak{p}$, and this is equivalent to the result stated above.

At this stage the notation will be changed. $\hat{F}_i$ will simply be written as F_i, and $\hat{k}$ as k. But it is to be understood that, in the new notation, k contains a subfield and $r(r-1)$ indeterminates u_{ij}, so that the phrase "independent of (u)" retains its meaning.

3. Projection of an isolated prime of (F). Assume that the co-ordinates have been chosen as indicated in §2, so that in particular the F_i are polynomials in x_n with highest coefficient unity and the other coefficients in R_{n-1}.

LEMMA 2. *Let $\mathfrak{p}$ be an isolated prime component of (F) and let $\bar{\mathfrak{p}} = R_{n-1} \cap \mathfrak{p}$ be its projection. Then $\bar{\mathfrak{p}}$ is an isolated prime component of (H).*

Proof. $\bar{\mathfrak{p}}$ is obviously a prime ideal in R_{n-1} containing (H); the essential point is to show that it occurs as an isolated component in a primary decomposition of (H).

Let $\mathfrak{o}$ be the quotient ring of R_n with respect to $\mathfrak{p}$, and $\bar{\mathfrak{o}}$ that of R_{n-1} with respect to $\bar{\mathfrak{p}}$. Let K be the residue class field $\bar{\mathfrak{o}}/\bar{\mathfrak{o}}\bar{\mathfrak{p}}$. This field can clearly be identified with a subfield of the residue class field $\mathfrak{o}/\mathfrak{o}\mathfrak{p}$. Also if $\xi_1, \xi_2, \ldots, \xi_n$ are the residue classes mod $\mathfrak{p}$ of $x_1, x_2, \ldots, x_n$, $\mathfrak{o}/\mathfrak{o}\mathfrak{p}$ is the field of fractions of the power series ring $k[[\xi_1, \ldots, \xi_n]]$, while a similar statement holds for K,

the element ξ_n being omitted. On the other hand ξ_n is algebraic over K, since it satisfies each of the equations $F_i = 0$, reduced mod $\mathfrak{p}$. And since the coefficients of the powers of x_n in each of the F_i are all of positive order, it follows that in the minimal equation of ξ_n over K all the coefficients, except the highest which is 1, will be power series in $\xi_1, \ldots, \xi_{n-1}$ of positive order. It follows that in any power series in $k[[\xi_1, \ldots, \xi_n]]$, a power of ξ_n greater than the degree of ξ_n over K can be replaced by lower powers without lowering the degree of the term in which it occurs. Each such power series can therefore be written as a polynomial in ξ_n over K. That is to say, the field $\mathfrak{o}/\mathfrak{o}\mathfrak{p}$ can be written as $K(\xi_n)$.

$\bar{\mathfrak{o}}$ and $\mathfrak{o}$ are both neighbourhood rings of irreducible algebroid varieties, and so are regular local rings **(2**, p. 33**)**. Their completions $\bar{\mathfrak{o}}^*$ and $\mathfrak{o}^*$ are therefore also regular, and so **(3**, p. 88**)** are isomorphic to power series rings in the appropriate number of indeterminates over their respective residue class fields, namely K and $K(\xi_n)$. For the present purpose it will be convenient to identify K and $K(\xi_n)$ with subfields of $\bar{\mathfrak{o}}^*$ and $\mathfrak{o}^*$, respectively, writing, in particular, $\bar{\mathfrak{o}}^* = K[[y_1, y_2, \ldots, y_{n-r}]]$. In this notation the maximal ideal $\bar{\mathfrak{o}}^*\mathfrak{p}$ of $\bar{\mathfrak{o}}^*$ is that generated by the y_i.

Now it has already been noted that ξ_n is a root of each of the equations $F_i = 0$, regarded as a polynomial equation in x_n with the coefficients reduced mod $\bar{\mathfrak{p}}$. But F_i, written as a polynomial in x_n, has coefficients in R_{n-1}, which is a subring of $\bar{\mathfrak{o}}^* = K[[y_1, y_2, \ldots, y_{n-r}]]$. Thus F_i is a polynomial in x_n with coefficients which are power series in the y_i over K such that, when the y_i are set equal to zero (this is equivalent to reducing mod $\bar{\mathfrak{p}}$) the resulting polynomial has ξ_n as a root. Also $\partial F_i/\partial x_n \neq 0$ for $x_n = \xi_n$ and all the y_j set equal to zero; for otherwise $\partial F_i/\partial x_n$ would be equal to zero mod $\mathfrak{p}$, contrary to the condition (2) made to hold in §2. It follows at once by the implicit function theorem for polynomial equations with power series coefficients that there is a power series ϕ_i in the y_j with coefficients in $K(\xi_n)$ and with constant term ξ_n such that $F_i(x_1, x_2, \ldots, x_{n-1}, \phi_i) = 0$.

The polynomials F_i in x_n have coefficients in $K[[y_1, y_2, \ldots, y_{n-r}]] = K[[y]]$. Let an algebraic extension of the field of fractions of this ring be made so that the F_i factorize completely into linear factors. Write

$$F_i = \prod (x_n - \phi_{ij}) \tag{1}$$

where, in particular $\phi_{i1} = \phi_i$ for each i. Then by the theory of the resultant of a pair of polynomials **(5)**

$$H_i = \prod (\phi_{ik} - \phi_{rj}) \qquad (i = 1, 2, \ldots, r-1). \tag{2}$$

Now the prime ideal $\bar{\mathfrak{p}}$ is independent of (u) (cf. §2) and $(H) \subset \bar{\mathfrak{p}}$. And so, by Lemma 1, the only common zeros of F_i and F_r for any i, these equations being reduced mod $\bar{\mathfrak{p}}$, must be common to all the F_i mod $\bar{\mathfrak{p}}$. But, since it has been arranged that only one component $\mathfrak{p}$ of (F) projects on $\bar{\mathfrak{p}}$ it follows at once that the only common zero of F_i and F_r reduced mod $\bar{\mathfrak{p}}$, for any i, is

ξ_n. And since ξ_n is a simple root of each F_i reduced mod $\bar{\mathfrak{p}}$ (by condition (2) of §2) the only factor of H_i which is congruent to zero modulo the appropriate extension of $\bar{\mathfrak{p}}$ is $\phi_i - \phi_r$. Thus (2) can be rewritten as

$$H_i = (\phi_i - \phi_r)K_i,$$

where K_i is not zero modulo a suitable extension of $\bar{\mathfrak{p}}$.

Now H_i is rational in the coefficients of F_i and F_r and so is in $K[[y]]$. On the other hand the ϕ_i are power series in the y_j with coefficients in $K(\xi_n)$, and so the same is true of the K_i. Writing for brevity $K' = K(\xi_n)$, this means that the K_i are in the ring $K'[[y]] = K'[[y_1, y_2, y_3, \ldots, y_{n-r}]]$ and are not zero modulo the maximal ideal $K'[[y]]\bar{\mathfrak{p}}$ of this ring. That is to say the K_i are units of $K'[[y]]$, and so

$$K'[[y]](H) = K'[[y]](\phi_1 - \phi_r, \phi_2 - \phi_r, \ldots, \phi_{r-1} - \phi_r). \tag{3}$$

Going back to equation (1), multiply out the factors on the right for which $j \neq 1$, and arrange the result as a power series in the y_i with coefficients in $K'[x_n]$. In particular write $A_i(x_n)$ for the term independent of the y_j. Thus (1) becomes:

$$F_i = (x_n - \phi_i)(A_i(x_n) + \ldots) \tag{4}$$

where the dots represent terms containing the y_j.

It has already been observed that ξ_n is a simple root of each of the F_i reduced mod $\bar{\mathfrak{p}}$, and so $A_i(\xi_n) \neq 0$ for each i. Now $\bar{\mathfrak{o}}^* \subset \mathfrak{o}^*$, and K' is the residue class field of $\mathfrak{o}^*$, and is identified with a subfield of this ring; finally $x_n \in \mathfrak{o}^*$. It follows that both factors on the right of (4) are in $\mathfrak{o}^*$. On the other hand, all the y_j are in $\bar{\mathfrak{p}}$ and so in $\mathfrak{o}^*\mathfrak{p}$, and $x_n - \xi_n \in \mathfrak{o}^*\mathfrak{p}$, whence the second factor on the right of (4) is congruent to $A_i(\xi_n)$ mod $\mathfrak{o}^*\mathfrak{p}$. Thus the second factor on the right of (4) is not zero mod $\mathfrak{o}^*\mathfrak{p}$, and so is a unit of $\mathfrak{o}^*$. It follows at once that

$$\begin{aligned} \mathfrak{o}^*(F) &= \mathfrak{o}^*(x_n - \phi_1, x_n - \phi_2, \ldots, x_n - \phi_r) \\ &= \mathfrak{o}^*(\phi_1 - \phi_r, \phi_2 - \phi_r, \ldots, x_n - \phi_r). \end{aligned} \tag{5}$$

Now compare equations (3) and (5). Since $\mathfrak{p}$ is an isolated prime component of (F), $\mathfrak{o}^*(F) = \mathfrak{o}^*\mathfrak{p}$, and so, by (5)

$$\mathfrak{o}^*\mathfrak{p} = \mathfrak{o}^*(\phi_1 - \phi_r, \phi_2 - \phi_r, \ldots, x_n - \phi_r).$$

A straightforward verification shows that the intersection of the ideal on the right of the last equation with $K'[[y]]$ is obtained simply by dropping the last generator. And so, applying equation (3), it follows that

$$\mathfrak{o}^*\mathfrak{p} \cap K'[[y]] = K'[[y]](H). \tag{6}$$

An element of $K'[[y]]$ is a power series in the y_j and is congruent mod $\mathfrak{o}^*\mathfrak{p}$ to its constant term, an element of K', namely the residue class field of the local ring $\mathfrak{o}^*$. It follows at once that $\mathfrak{o}^*\mathfrak{p} \cap K'[[y]] \subset K'[[y]]\bar{\mathfrak{p}}$ .The reverse inclusion relation is obvious. Hence (6) is equivalent to

$$K'[[y]](H) = K'[[y]]\bar{\mathfrak{p}}. \tag{7}$$

The next step is to form the intersection of each side of the last equation with $K[[y]]$. Noting that K' is a finite algebraic extension of K, and writing the elements of K' in terms of a linear basis (including the element 1) over K it follows at once from (7) that $K[[y]](H) = K[[y]]\bar{\mathfrak{p}}$. Since $K[[y]] = \bar{\mathfrak{o}}^*$ this implies that $\bar{\mathfrak{p}}$ is an isolated prime component of (H) as required.

4. Equations over a field with a valuation. Let K be a field complete with respect to a valuation v, the value group being written additively. Thus $v(ab) = v(a) + v(b)$ and $v(a + b) \geqslant \min(v(a), v(b))$, where a and b are any elements of K. Extend v to the algebraic closure $\bar{K}$ of K; this can be done in a unique manner **(5)**. The extended valuation will still be denoted by v.

For each α in the value group of v let N_α be the set of elements of $\bar{K}$ with values greater than α. Then the collection of sets of the type N_α can be taken as the basis of the neighbourhoods of 0 defining on $\bar{K}$ the structure of a topological group under addition. With this topology $\bar{K}$ is actually a topological field; that is to say, the operation of multiplication is also continuous.

LEMMA 3. *Let $F(z)$ and $F'(z)$ be two polynomials in z of the same degree, both with highest coefficient* 1 *and with all other coefficients in K having non-negative values under v. Let V be a preassigned neighbourhood of* 0 *in $\bar{K}$. Then if the coefficients of F' are sufficiently near those of F, in the sense of the topology just defined, each root of F' will differ from some root of F by an element of V.*

Proof. According to the definition of extended valuations (cf. van der Waerden **(5)** the conditions on the coefficients of F' imply that $v(\zeta_i') \geqslant 0$ for each root ζ_i' of F'. Let c be the smallest of the values under v of the differences of corresponding coefficients of F and F'. Then, for each root of F', the definition of a valuation along with the condition $v(\zeta_i') > 0$ implies that $v(F(\zeta_i') - F'(\zeta_i')) \geqslant c$. That is to say $v(F(\zeta_i')) > c$. But if $\zeta_1, \zeta_2, \ldots, \zeta_m$ are the roots of F,

$$F(\zeta_i') = (\zeta_i' - \zeta_1)(\zeta_i' - \zeta_2)\ldots(\zeta_i' - \zeta_m)$$

and so the last inequality implies that, for some j, $v(\zeta_i' - \zeta_j) > c/m$. If the preassigned neighbourhood V of 0 is assumed to be the set of elements of $\bar{K}$ for which $v(a) > c/m$, the last inequality establishes the lemma.

5. Lifting theorem for analytic equivalence. Let (F) and (F') be ideals in $k[[x_1, x_2, \ldots, x_n]]$ with sets of generators $F_1, F_2, \ldots, F_r$ and $F_1', F_2', \ldots, F_r'$, respectively. The object here is to show that, under certain conditions, if the orders of the $F_i - F_i'$ are high enough there exists an analytic equivalence between the isolated prime components of (F) and certain components of (F'). Clearly the final result is not going to be affected if the various adjustments described in §2 are made in advance. In particular the F_i can be assumed to have been replaced by polynomials in x_n. The complementary remark to the Weierstrass Preparation Theorem in §2 implies that the F_i' can also be replaced by polynomials in x_n, and that F_i' will be of the same degree as F_i,

the differences of corresponding coefficients being of high order if the order of $F_i' - F_i$ is high enough. It will also be assumed that the F_i and F_i' have been replaced in advance by generic linear combinations involving indeterminates u_{ij} as in §2, so that the results of §3 can be applied; but as at the end of §2, the presence of the u_{ij} will not be indicated in the notation.

Let $H_1, H_2, \ldots, H_{r-1}$ be the set of resultants of F_r with $F_1, F_2, \ldots, F_{r-1}$, respectively, with respect to x_n, and let $H_1', H_2', \ldots, H_{r-1}'$ be calculated in the same way from the F_i'. Since the resultant of two polynomials is rational in the coefficients in the polynomials, it follows that the orders of the $H_i' - H_i$ will be arbitrarily high if those of the $F_i' - F_i$ are high enough.

Assume that the following condition, to be referred to later as condition **A**, holds; it will be shown later that this is certainly so for the case in which the variety defined by (F) is a curve:

A. Given an integer m, there exists an integer m' such that, if the orders of the $F_i - F_i' > m'$, then there is an analytic equivalence S in R_{n-1} carrying the isolated prime components of (H) into components of (H'), and also an extension of this to an analytic equivalence T in R_n carrying (G, F_r) into (G', F_r'), where (G) is the intersection of the isolated primes of (H) and (G') is its image under S, and S and T are both of order $> m$.

THEOREM 1. *If the integers m and m' of condition* **A** *are big enough, then the analytic equivalence T carries the isolated prime components of (F) into components of (F').*

Proof. Let the generators of (G) be denoted by G_i, $i = 1, 2, \ldots, q$, those of (G') by G_i', $i = 1, 2, \ldots, q$. In the following proof (x) will stand for the set $(x_1, x_2, \ldots, x_{n-1})$, and $S^{-1}(x)$ for the set of series $S^{-1}(x_i)$ obtained by applying the inverse S^{-1} of S to the x_i. $S^{-1}(\xi)$ will denote the result of replacing the x_i in $S^{-1}(x)$ by their residue classes mod $\mathfrak{p}$, where $\mathfrak{p}$ is a given isolated prime component of (F). Condition **A** states that T carries (G, F_r) into (G', F_r'), and, of course, T^{-1} effects the reverse transformation. Hence there are elements A_i and B of R_n such that

$$T^{-1}F_r'(x, x_n) = \sum A_iG_i(x) + BF_r(x, x_n),$$

or, what is the same thing,

$$F_r'(S^{-1}(x), T^{-1}(x_n)) = \sum A_iG_i(x) + BF_r(x, x_n). \tag{8}$$

By Lemma 2, $\bar{\mathfrak{p}} = R_{n-1} \cap \mathfrak{p}$ is an isolated prime component of (H) and so of (G); from which it follows that $G_i(\xi) = 0$ for each i. Substituting the ξ_i for the x_i, $i = 1, 2, \ldots, n-1$, in (8), and writing $(\xi') = S^{-1}(\xi)$:

$$F_r'(\xi', T^{-1}(x_n)) = B(\xi, x_n)F_r(\xi, x_n).$$

Since, however, ξ_n is a root of $F_r(\xi, x_n) = 0$, the last equation implies that $\xi_n' = T^{-1}(\xi_n)$ is a root of $F_r'(\xi', x_n) = 0$. The main task of this proof is to

show that, under the conditions of the theorem, ξ_n' is a root of each of the equations $F_i'(\xi', x_n) = 0$. This will be done now by applying Lemma 3.

As in §3 take K as the residue class field of the local ring $\bar{\mathfrak{o}}$. A valuation is to be introduced on K in such a way that the ξ_i, $i = 1, 2, \ldots, n - 1$, all have values greater than zero. This can be done as follows. Making, if necessary, a suitable linear change of variables (it will be assumed that this has already been done in advance along with the other adjustments in §2) it can be arranged that no element of $k[[x_1, x_2, \ldots, x_{n-r}]]$ vanishes when reduced mod $\bar{\mathfrak{p}}$, while for each i from 1 to $r - 1$, ξ_{n-r+i} is integral over $k[[\xi_1, \xi_2, \ldots, \xi_{n-r+i-1}]]$. In addition, the coefficients of the minimal equation of ξ_{n-r+i} over the ring $k[[\xi_1, \xi_2, \ldots, \xi_{n-r+i-1}]]$, except the first which is 1, are all power series of positive order in the ξ_j. The assertions just made follow from repeated applications of the Weierstrass Preparation Theorem. Define the valuation v on the field of fractions of $k[[\xi_1, \xi_2, \ldots, \xi_{n-r}]]$ by setting $v(\alpha)$, for a power series α in the ξ_i, equal to the order of α. This valuation v can then be extended in the usual way, step by step, to K, and the nature of the coefficients of the minimal equations of the ξ_i ensures that $v(\xi_i) > 0$ for all i. The field of fractions of $k[[\xi_1, \xi_2, \ldots, \xi_{n-r}]]$ is complete with respect to v, and so, since K is obtained by a finite algebraic extension, K is complete with respect to the extended valuation **(5)**. As in §4 let $\bar{K}$ be the algebraic closure of K and let v be extended to $\bar{K}$.

The equations $F_i(\xi, x_n) = 0$ and $F_i'(\xi', x_n) = 0$ are now to be compared. Note that, since $(\xi') = S^{-1}(\xi)$, K is the field of fractions of $k[[\xi_1', \xi_2', \ldots, \xi_{n-1}']]$ and so the equations to be examined both have their coefficients in K. If the order of $F_i(x, x_n) - F_i'(x, x_n)$ and that of the analytic equivalence S are high enough it is clear that $F_i(\xi, x_n)$ and $F_i'(\xi', x_n)$ will be of the same degree in x_n and that the differences of corresponding coefficients will have arbitarily high values under v. In particular the highest coefficients will both be 1, and all the other coefficients will have non-negative values. Conditions are therefore suitable for the application of Lemma 3.

By Lemma 3, if the integers m and m' of condition **A** are large enough, each root of $F_i'(\xi', x_n) = 0$ is arbitrarily near some root of $F_i(\xi, x_n) = 0$. Choose the neighbourhood V of Lemma 3 so that, among the set of all the roots of $F_i(\xi, x_n) = 0$ and $F_r(\xi, x_n) = 0$, the common root ξ_n being counted just once, no two differ by an element of $V + V$. Then if m and m' are big enough, each root of $F_i'(\xi', x_n) = 0$ is in a V-neighbourhood of some root of $F_i(\xi, x_n) = 0$; a similar statement holds for $F_r'(\xi', x_n) = 0$ in relation to $F_r(\xi, x_n) = 0$. It follows that no root of $F_i'(\xi', x_n) = 0$ can coincide with a root of $F_r'(\xi', x_n) = 0$ except possibly roots of these equations lying in a V-neighbourhood of ξ_n.

But, by hypothesis, S carries $\bar{\mathfrak{p}}$ into some component of (H'). And so, since $(\xi') = S^{-1}(\xi)$, (ξ') is a zero of (H'). In particular the resultant of F_i' and F_r' with respect to x_n vanishes when (x) is replaced by (ξ'), whence the equations $F_i'(\xi', x_n) = 0$ and $F_r'(\xi', x_n) = 0$ must have at least one common root. It has just been shown that this common root must be in a V-neighbourhood

of ξ_n; if it is proved that $F_r'(\xi', x_n) = 0$ has only one root near ξ_n when the coefficients of $F_r'(\xi', x_n)$ are near those of $F_r(\xi, x_n)$ then this root must be ξ_n', which, being equal to $T^{-1}(\xi_n)$, is certainly near ξ_n if m' is large. The fact that ξ_n' is a root of $F_i'(\xi', x_n) = 0$ would then be established as required.

To check this last point note that if, on the contrary, arbitrary neighbourhoods of ξ_n contain two roots of $F_r'(\xi', x_n) = 0$ for large values of m and m', then in the limit as m, m' tend to infinity, it would turn out that ξ_n would be a double root of $F_r(\xi, x_n) = 0$. This is not so, and thus the proof that ξ_n' is a root of each $F_i'(\xi', x_n) = 0$ is completed.

The result just obtained implies that $(\xi_1', \xi_2', \ldots, \xi_n')$ is a zero of the ideal (F'), and so of some prime component $\mathfrak{p}'$ of (F'). Now if $\phi \in \mathfrak{p}'$, $\phi(\xi', \xi_n') = \phi(S^{-1}(\xi), T^{-1}(\xi_n)) = 0$; hence $T^{-1}\phi \in \mathfrak{p}$ and so $T^{-1}\mathfrak{p}' \subset \mathfrak{p}$. A similar argument shows that $T\mathfrak{p} \subset \mathfrak{p}'$. Hence $\mathfrak{p}' = T\mathfrak{p}$. That is to say, it has been shown that T carries $\mathfrak{p}$ into a component of (F'). Since this holds for any isolated prime component of (F) the proof of the theorem is completed.

6. Algebroid curves. Attention will now be restricted to the case $r = n - 1$, the components of $(F) = (F_1, F_2, \ldots, F_{n-1})$ being all one-dimensional, and the result indicated in the introduction will be proved. As pointed out there, the proof will be by induction on n, the case $n = 2$ being established by means of the following theorem of Samuel (stated here only in the case of two variables):

LEMMA 4. *If F and F' are power series in x and y over a field k and if $F - F'$ is in the ideal $(x, y)(\partial F/\partial x, \partial F/\partial y)^2$, then there is an analytic equivalence of $k[[x, y]]$ carrying F into F'.*

Proof. See Samuel **(4)**.

Now if F has no multiple factors, F, $\partial F/\partial x$ and $\partial F/\partial y$ have an isolated common zero at the origin, and so a power of the ideal (x, y) is in $(\partial F/\partial x, \partial F/\partial y)$ mod F. It follows easily **(4)** that:

LEMMA 5. *If $F - F'$ is of sufficiently high order and F is free of multiple factors then the principal ideals (F) and (F') are analytically equivalent.*

In the case where F does possibly have double factors, let G be the product of the simple factors of F. Thus G is a product of simple factors. In order that Lemma 5 can be applied to this situation, it must be shown that F' has a factor G' differing from G by terms of high degree, provided that the order of $F - F'$ is high enough. This will be done by means of the following modification of Hensel's Lemma.

LEMMA 6. *Let F, G, H be polynomials in y with coefficients which are power series in x over a field k, and let $F = GH$. Also suppose that G and H have no common factor. Then if F' is a polynomial in y of the same degree as F, also with*

power series in x as coefficients, and if the differences of corresponding coefficients are of sufficiently high order, in particular the coefficients of the highest powers of y in F, F', G, and H all being 1, *F' will have a factorisation $G'H'$ where G' and H' are polynomials in y of the same degrees as G and H respectively, and such that $G - G'$ and $H - H'$, written as series in x, are of arbitrarily high order. The highest coefficients of G' and H' will be* 1.

Proof. If the orders of the differences of corresponding coefficients of F and F' are all greater than r, it will be convenient to use the notation $F \equiv F'$ (x^r). This notation will be used throughout this proof.

By hypothesis, the highest common factor of G and H, regarded as polynomials in y over the field of fractions of the power series ring $k[[x]]$, is 1. Remembering that an element of this field of fractions can be written as a series of positive powers of x divided by a power of x, it follows at once that there is an integer h, and polynomials A and B in y with coefficients in $k[[x]]$ such that the degrees of A and B in y are less than those of H and G, respectively, and

$$AG + BH = x^h. \tag{9}$$

Now let s be any integer greater than $2h$, and suppose that $F \equiv F'(x^s)$. The lemma will be proved if it can be shown that, for each $q > s$, there are polynomials G_q and H_q in y with coefficients in $k[[x]]$, the highest coefficients being in each case 1, of the same degrees in y as G and H respectively, and satisfying the conditions

$$\left.\begin{aligned} F' &\equiv G_qH_q(x^q) \\ G_q &\equiv G(x^{s-h}) \\ H_q &\equiv H(x^{s-h}) \end{aligned}\right\}. \tag{10}$$

For then G' and H' can be taken as the limits of G_q and H_q as q tends to ∞. This result will be proved by induction on q; it is clearly true for $q = s$, taking $G_s = G$ and $H_s = H$. Suppose that G_q and H_q have already been found satisfying (10). It will now be shown that, setting $G_{q+1} = G_q + ux^{q-h}$ and $H_{q+1} = H_q + vx^{q-h}$, u and v can be determined as polynomials in y over $k[[x]]$ of degrees less than those of G and H respectively in such a way that conditions (10) hold with q replaced by $q + 1$.

By the first of the conditions (10) $F' = G_qH_q + wx^q$, where w is a polynomial in y over $k[[x]]$, the degree in y being less than that of F. And so

$$F' - G_{q+1}H_{q+1} = x^{q-h}(wx^h - vG_q - uH_q) - uvx^{2q-2h}. \tag{11}$$

Multiply (9) by w, obtaining $AwG + BwH = wx^h$. Here the right-hand side is of degree in y less than that of F, and so the standard adjustment, using the long division algorithm, can be made to Aw and Bw, replacing them by polynomials A' and B' in y over $k[[x]]$ of degrees less than those of H and G respectively. Thus

$$A'G + B'H = wx^h.$$

Applying the second and third conditions of (10)

$$(12) \qquad A'G_q + B'H_q \equiv wx^h(x^{q-h}).$$

Now in the definitions of G_{q+1} and H_{q+1} take $u = A'$ and $v = B'$, and (11) along with (12) gives at once the result $F' - G_{q+1}H_{q+1} \equiv 0$ (x^{q+1}). The other two conditions corresponding to (10) with q replaced by $q + 1$ are clearly satisfied and so the induction is completed, and with it the proof of this lemma.

The above lemmas can now be combined to give the following result:

LEMMA 7. *Let F and F' be power series in x and y over k. Then if the order of $F - F'$ is sufficiently high there is an analytic equivalence in $k[[x, y]]$ which carries the isolated prime components of the ideal (F) into components of (F').*

Proof. It may be assumed that co-ordinates have been changed so that F and F' can be replaced by polynomials in y over $k[[x]]$ of the same degree in y and having highest coefficient 1. Then the isolated prime components of (F) are the components of (G) where G is the product of the simple factors of F. By Lemma 6 there is a factor G' of F' such that the order of $G - G'$ is high if that of $F - F'$ is high enough. Applying Lemma 5 to G and G', the result follows at once.

The main result of this paper can now be stated:

THEOREM 2. *Let an algebroid curve in n-space be defined by the ideal $(F) = (F_1, F_2, \ldots, F_{n-1})$ in $k[[x_1, x_2, \ldots, x_n]]$. Then if the orders of the series $F_i - F_i'$ are high enough, there is an analytic equivalence of arbitrarily high order carrying the isolated prime components of (F) into components of the ideal (F').*

Proof. As usual the preliminary adjustments to the co-ordinates and to the generators of (F) described in §2 will be assumed to have been carried out in advance, so that the results of §§3, 5 can be applied here. The proof of the present theorem will be carried out by induction on n, the case $n = 2$ having been already established in Lemma 7. Clearly the proof will be completed if it is shown that, on the basis of the induction hypothesis that the present theorem holds for algebroid curves in $(n - 1)$-space, condition **A** of §5 must hold; for then Theorem 1 can be applied to give the transition from $n - 1$ to n. But the holding of condition **A** under this induction hypothesis was proved as Lemma 6.1 in **(6)**. Admittedly I was dealing in that paper with algebroid curves defined over the real field, but the proof of the quoted lemma was entirely algebraic in character, and so applies equally well to the present situation. The inductive proof of Theorem 2 is thus completed.

7. Projection of analytically equivalent curves. It has already been shown that if C and C' are analytically equivalent curves in $(n - 1)$-space

then they lift into analytically equivalent curves in a hypersurface in n-space, provided that the given analytic equivalence is of sufficiently high order cf **(6**, Lemma 6.1**)**. A sort of converse to this will now be obtained, namely, that algebroid curves of n-space which are analytically equivalent to a sufficiently high order will project into analytically equivalent curves of $(n-1)$-space. This result is non-trivial since a curve and its projection need not be analytically equivalent; for example the space curve $x = t^3$, $y = t^4$, $z = t^5$ is not analytically equivalent to any plane curve. And it is certainly not obvious that a given analytic equivalence can be modified in such a way that series not involving x_n are carried into series not involving x_n.

THEOREM 3. *Let C and C' be algebroid curves in n-space analytically equivalent under T, and, possibly after a sufficiently general change of co-ordinates, let $\bar{C}$ and $\bar{C}'$ be their projections into $(x_1, x_2, \ldots, x_{n-1})$-space. Then if T is of sufficiently high order, $\bar{C}$ and $\bar{C}'$ will be analytically equivalent to an arbitrarily high order.*

Proof. Let $\mathfrak{A}$ be the ideal of C, $\mathfrak{A}'$ that of C'. Of course, by the definition of algebroid varieties (see **(2)**) C is actually defined by some ideal in R_n; $\mathfrak{A}$ is to be the radical of that ideal. Thus $\mathfrak{A}$ is an intersection of primes, say $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_m$. A similar remark holds for $\mathfrak{A}'$. Write $\mathfrak{p}_i' = T(\mathfrak{p}_i)$, $i = 1, 2, \ldots, m$.

The first step of the proof is to find an ideal (F) in R_n having exactly $n-1$ generators and having $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_m$ as isolated prime components. Let $\mathfrak{o}_i$ be the quotient ring of R_n with respect to the prime ideal $\mathfrak{p}_i$. Then $\mathfrak{o}_i$ is the neighbourhood ring of a variety and so is a regular local ring **(2**, p. 33**)**. The maximal ideal of $\mathfrak{o}_i$ is $\mathfrak{o}_i\mathfrak{p}_i$. For each i, j with $i \neq j$ choose ϕ_j^i in $\mathfrak{p}_i$ but not in $\mathfrak{p}_j$; this is possible since no two of the $\mathfrak{p}_i$ contain one another. Then make the definition

$$\phi_j = \prod_{i \neq j} \phi_j^i$$

the product being taken over i. It is clear that ϕ_j is in each $\mathfrak{p}_i$ for $i \neq j$, but is not in $\mathfrak{p}_j$, since none of its factors is. Next define ψ_i by

$$\psi_i = \sum_{j \neq i} \phi_j$$

the summation being over j. ψ_i is in $\mathfrak{p}_i$, since each of its summands is, but $\psi_i \equiv \phi_j\ (\mathfrak{p}_j)$, and so $\psi_i \notin \mathfrak{p}_j$, for $i \neq j$. Setting $\psi_i^{n-1} = \psi_i$ choose co-ordinates so that the $\mathfrak{p}_i$ have distinct projections in R_{n-1}, and, proceeding as above, find an element ψ_i^{n-2} of $\mathfrak{p}_i \cap R_{n-1}$ which is not in any $\mathfrak{p}_j \cap R_{n-1}$ for $i \neq j$. In this way, step by step, a set of elements $\psi_i^1, \psi_i^2, \ldots, \psi_i^{n-1}$ of $\mathfrak{p}_i$ is obtained with the property that they are not in any $\mathfrak{p}_j$ for $j \neq i$. In addition, if $\xi_1, \xi_2, \ldots, \xi_n$ denote the residue classes of $x_1, x_2, \ldots, x_n$ mod $\mathfrak{p}_i$, it is known that, possibly after a suitable linear change of co-ordinates $\xi_2, \xi_3, \ldots, \xi_n$ are separably algebraic over the field of fractions of $k[[\xi_1]]$ **(2**, p. 32**)**. It follows that, possibly after discarding superfluous factors, one can assume that

$$\text{(13)} \qquad \partial\psi_i^j/\partial x_{j+1} \not\equiv 0(\mathfrak{p}_i).$$

Finally define

$$F_j = \prod_i \psi_i^j.$$

Then the ideal (F) generated by the F_i has the above asserted property, namely that the $\mathfrak{p}_i$ are all isolated prime components. For, extending to $\mathfrak{o}_i$, $\mathfrak{o}_i(F) = \mathfrak{o}_i(\psi_i^1, \psi_i^2, \ldots, \psi_i^{n-1})$, all the other factors of the F_j being units of $\mathfrak{o}_i$. The condition (13) implies that the ψ_i^j form a regular system of parameters in $\mathfrak{o}_i$ **(2**, p. 34**)** and so $\mathfrak{o}_i(F) = \mathfrak{o}_i\mathfrak{p}_i$; that is to say, $\mathfrak{p}_i$ is an isolated prime component of (F) as was to be shown.

Let the given analytic equivalence T carry F_i into F_i', and write (F') for the ideal generated by the F_i' in R_n.

If necessary making a further linear change of co-ordinates, apply the Weierstrass Preparation Theorem to the F_i and the F_i'. The ideals (F) and (F') can thus be generated by $G_1, G_2, \ldots, G_{n-1}$ and $G_1', G_2', \ldots, G_{n-1}'$ respectively, where the G_i and G_i' are all polynomials in x_n. Also if the order of T is high enough, the orders of the corresponding coefficient differences of G_i and G_i' for each i can be made arbitrarily high. The top coefficients of the G_i and G_i' are of course all equal to 1. It will in addition be assumed that the procedure of §2 has been applied to the G_i and G', introducing indeterminates u_{ij}, whose presence, however, is not indicated by the notation.

Let H_i be the resultant with respect to x_n of G_i and G_{n-1}, and let (H) denote the ideal generated in R_n by the H_i; the H_i' and (H') are to be similarly defined from the G_j'. Since the resultants of polynomials are rational in the coefficients, it follows that the order of $H_i - H_i'$, for each i, can be made arbitrarily high if the order of T is high enough.

From the last remark it follows by means of Theorem 2 that if the order of T is high enough there will be an analytic equivalence S in R_{n-1} of arbitrarily high order carrying the isolated prime components of (H) into components of (H'). In particular, for each i, $\bar{\mathfrak{p}}_i = \mathfrak{p}_i \cap R_{n-1}$ is an isolated prime component of (H), provided the co-ordinates are suitably chosen (Lemma 2). Thus $S(\bar{\mathfrak{p}}_i)$ is a component $\bar{\mathfrak{p}}_i'$ of (H'). It is required to prove now that $\bar{\mathfrak{p}}_i' = \mathfrak{p}_i' \cap R_{n-1}$. If this is known for each i, then it will be known that S carries $\bar{C}$ into $\bar{C}'$ as was to be proved.

Write ξ_j for the residue class of x_j mod $\mathfrak{p}_i$, and denote $(\xi_1, \xi_2, \ldots, \xi_{n-1})$ by (ξ). Thus (ξ) is a zero of $\bar{\mathfrak{p}}_i$. In the notation employed in the proof of Theorem 1, it follows that $S^{-1}(\xi)$ is a zero of $\bar{\mathfrak{p}}_i'$. But $\bar{\mathfrak{p}}_i'$ is a component of (H'), and so, for some value of j which is to be fixed in the meantime (any value of course will do) $S^{-1}(\xi)$ is a zero of H_j'. It follows that $G_j'(S^{-1}(\xi), x_n)$ and $G_{n-1}'(S^{-1}(\xi), x_n)$, polynomials in x_n, have a common zero. Call this zero ξ_n'.

The polynomials $G_j'(S^{-1}(\xi), x_n)$ and $G_j(\xi, x_n)$ are now to be compared. Assume as in the proof of Theorem 1 a valuation has been introduced on the field of fractions of $k[[\xi_1, \xi_2, \ldots, \xi_{n-1}]]$ and extended to its algebraic closure. If the order of T, and so that of S, is sufficiently high the values of the differ-

ences of the corresponding coefficients of $G_j'(S^{-1}(\xi), x_n)$ and $G_j(\xi, x_n)$ will be arbitrarily large. Lemma 3 then shows that, if the order of T, and so of S, is large enough, the root ξ_n' of $G_j'(S^{-1}(\xi), x_n)$ will lie in a V-neighbourhood of some root of $G_j(\xi, x_n)$, where V is preassigned. Similarly, replacing j by $n-1$, ξ_n' will lie in a V-neighbourhood of some root of $G_{n-1}(\xi, x_n)$. If V is taken so that no two of the roots of $G_j(\xi, x_n)$ and $G_{n-1}(\xi, x_n)$ differ by an element of $V+V$, this implies that ξ_n' is in a V-neighbourhood of a common root of these polynomials. In view of Lemma 1 the only common root of $G_j(\xi, x_n)$ and $G_{n-1}(\xi, x_n)$ is common to all the $G_h(\xi, x_n)$, $h = 1, 2, \ldots, n-1$, and, since condition (3) of §2 is supposed to be satisfied here by the G_h, the only such common root is ξ_n. Thus if the order of T is sufficiently high ξ_n' is arbitrarily near ξ_n. It follows that ξ_n' is the unique common root of all the $G_{h'}(S^{-1}(\xi), x_n)$, $h = 1, 2, \ldots, n-1$, unique since otherwise a limiting procedure would lead to a double root of $G_{n-1}(\xi, x_n)$, which is ruled out by condition (2) of §2. Thus $(S^{-1}(\xi), \xi_n')$ is a zero of a uniquely defined prime component $\mathfrak{p}_i''$ of $(F') = (G')$. The proof of the theorem will be completed by showing that $\mathfrak{p}_i'' = \mathfrak{p}_i'$.

The proof that $\mathfrak{p}_i'' = \mathfrak{p}_i'$ will involve an additional technical device which will now be described. Let $\mathfrak{p}$ be any prime ideal of R_n of dimension one. Then, as has already been remarked, the fraction field of $R_n/\mathfrak{p}$ is an algebraic extension of a field of power series in one variable. Thus it is a field with a discrete valuation and clearly contains a congruent representative of the residue class field corresponding to this valuation, namely k itself. Thus, by a known theorem of valuation theory, the field of fractions of $R_n/\mathfrak{p}$ can be identified with a subfield of the field of fractions of $k[[t]]$. This could also be expressed by saying that the algebroid curve defined by $\mathfrak{p}$ can be parametrized by means of power series in t.

Now if parametrizations are introduced as above on a number of algebroid curves, the same parameter symbol t can be used for all of them. This gives a means of comparing different curves. An immediate question which arises is: What is the relation between different parametrizations of the same curve? The answer is that they may be obtained from one another by means of an invertible power series substitution, replacing t by a power series in t having zero constant term but non-zero linear term. This is easily seen by noting that in a parametrization, the parameter can be identified with any element of minimum value.

Let $\mathfrak{p}_1$ and $\mathfrak{p}_2$ be two ideals of R_n and let $x_i^1(t)$ and $x_i^2(t)$, i running in each case from 1 to n, be parametrizations of the corresponding curves. Define

$$\delta(\mathfrak{p}_1, \mathfrak{p}_2) = \sup[\text{min order of } [x_i^1(t) - x_i^2(t)]]$$

the supremum being taken over all possible parametrizations of the two curves, and the minimum over i; the order of an element is to be its order as a series in t.

In the first place note that $\delta(\mathfrak{p}_1, \mathfrak{p}_2)$ is finite for $\mathfrak{p}_1 \neq \mathfrak{p}_2$. To prove this, it

is sufficient in the above definition to assume that the parametrization of the first curve is kept fixed; this can always be arranged by means of a simultaneous change of parameter without changing the order of any of the $x_i^1(t) - x_i^2(t)$. But then, if the orders of the $x_i^1(t) - x_i^2(t)$ were unbounded, the parametrization of the second curve being varied, it would follow by a limiting process that the two curves would have a common parametrization, contrary to the fact that $\mathfrak{p}_1 \neq \mathfrak{p}_2$.

Returning now to the proof that $\mathfrak{p}_i'' = \mathfrak{p}_i'$, apply T^{-1} to both ideals. Thus it is to be shown that $T^{-1}\mathfrak{p}_i'' = \mathfrak{p}_i$. To do this replace the zero (ξ, ξ_n) of $\mathfrak{p}_i$ by some parametrization. The zero $(TS^{-1}(\xi), T(\xi_n'))$ of $T^{-1}\mathfrak{p}_i''$ is automatically replaced by a parametrization of the appropriate curve, simply by substitution of the power series. And the orders (in the parameter t) of the corresponding co-ordinate differences of (ξ, ξ_n) and $(TS^{-1}(\xi), T(\xi_n'))$ will be arbitrarily high if the orders of T and S are high enough (it will be remembered that the ξ_j are all of positive value, as noted in the proof of Theorem 1, and also that the value of $\xi_n' - \xi_n$ is high when the orders of S and T are high enough). It follows at once that $\delta(\mathfrak{p}_i, T^{-1}\mathfrak{p}_i'')$ can be made arbitrarily large if the order of T is high enough. Then choose T and S so that $\delta(\mathfrak{p}_i, T^{-1}\mathfrak{p}_i'') > \delta(\mathfrak{p}_i, \mathfrak{p}_j)$ for all components $\mathfrak{p}_j$ of (F). Then $T^{-1}\mathfrak{p}_i''$, which is a component of (F), can only be equal to $\mathfrak{p}_i$, as was to be proved.

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INVOLUTIONS ASSOCIATED WITH THE BURKHARDT CONFIGURATION IN [4]

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1. Introduction. Horadam **(11)** has established the existence of a locus L in [8] (projective 8-space) having order 45 and dimension 4, which is invariant under a group of order 51840×81 (the Clifford similarity transform group CT). Associated with CT are two other groups, the Clifford collineation group CG of order 81, and the Clifford substitution group CS of order 51840. Furthermore, CS may be regarded as either a subgroup of CT, or a symplectic group of index matrices of size 4. Among the matrices of size 9 which perform the operations of CT, there is a set of 81 involutory, symmetric, orthogonal matrices JW. As collineation matrices in [8], these produce 81 pairs of invariant spaces Σ, Π of dimensions 3 and 4 respectively. These [4]'s give rise to a configuration **C** invariant under the operations of CT, consisting of 360 points, 1080 lines, 120 Jacobian planes, and 81 [4]'s, and their various inter-relationships.

Familiarity with the theory and notation of **(11)** is assumed in this paper.

If a section of L is taken by the Π-space whose determining equations are $x_{ij} = x_{2i2j}$ $(i, j = 0, 1, 2)$, the resulting locus consists of just 45 points forming a Clifford-derived configuration identical with the Burkhardt configuration in [4], **B**, which is associated with the rational Burkhardt quartic primal. Such points (nodes in **B**) have co-ordinates of two types, namely,

Type I: $(-2\ \epsilon^t\ \epsilon^u\ \epsilon^v\ \epsilon^w)$ with $t + u + v + 2 \equiv 0 \pmod 3$.

Type II: $(.\ ;\ 1\ \ -\epsilon^a\ .\ .)$.

Twenty-seven of these nodes belong to Type I and the remaining 18 to Type II. A general node of Type I will, for convenience, be labelled P_{tuvw}, or merely P_t, where there is no possible ambiguity, while the given node of Type II will be designated by P_a.

Comparison may be made between our Clifford-derived configuration, identical with **B**, and the figure in [4] which Edge **(6)** has explained in detail. Based on $GF(3)$, this latter figure differs fundamentally from **B**, but, at the same time, displays striking similarities.

2. Involutions concerning nodes and their Jordan primes. Invariance of **B** is preserved by the harmonic inversion, or projection $p(A)$, with respect to any node A and its Jordan prime. This projection is the operation of the group $\frac{1}{2}CS$ (subgroup of CS of index 2 and also the cubic surface (sub)group) which leaves invariant A and the 12 nodes in the Jordan prime of A, and

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which interchanges in pairs the two nodes, other than A, on each of the 16 κ-lines through A. Thus $p(A)$ is involutory. (Jordan primes in the Π lie on L and do not intersect any of the 81 solids Σ.) Operations relating to nodes P_t and P_a will usually be denoted by $p(t)$ and $p(a)$ respectively. Explicit matrix forms for the involutory projections $p(t)$ have been obtained in **(11)**, wherein the two corresponding involutory matrices in [8], $p^*(t)$ and $p'^*(t) = Jp^*(t)$, which induce identical collineations in Π, and the associated involutory symplectic index matrices $\mathbf{p}(t)$ and $\mathbf{p}'(t) = 2\mathbf{p}(t)$ are also found. (Starred notation used in **(11)** has been altered here for convenience.)

For the 18 Burkhardt nodes of Type II, the related matrices, which are not given in **(11)**, are much less complicated. Since they fall into six similar triads, there is nothing lost in concentrating on only one triad, namely P_a $(a = 0, 1, 2)$. We deduce that

$$\mathbf{p}(a) = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \epsilon^{2a} & \cdot & \cdot \\ \cdot & \epsilon^{a} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}.$$

Correspondingly, in [8], $p^*(a)$ has only one non-zero element in each row and column, these being: unity in positions $\alpha_{00}{}^{00}$, $\alpha_{11}{}^{11}$, $\alpha_{21}{}^{12}$, $\alpha_{12}{}^{21}$, $\alpha_{22}{}^{22}$; ϵ^a in positions $\alpha_{10}{}^{01}$, $\alpha_{20}{}^{02}$; and ϵ^{2a} in positions $\alpha_{10}{}^{10}$, $\alpha_{02}{}^{20}$. Of course, $p'^*(a) = Jp^*(a)$. Index matrices associated with these are

$$\mathbf{p}(a) = \begin{bmatrix} \cdot & \cdot & 1 & 2a \\ \cdot & \cdot & \cdot & 1 \\ 1 & a & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \end{bmatrix}$$

of which the elements belong to the finite field $GF(3)$. The basic Clifford set is easily shown to be converted by $\mathbf{p}(0)$, $\mathbf{p}(1)$, $\mathbf{p}(2)$ into the Clifford sets

$$\begin{matrix} W^{01}_{00}, & W^{01}_{01}, & W^{10}_{01}, & W^{10}_{11}, & W^{00}_{11}, \\ W^{01}_{00}, & W^{02}_{01}, & W^{11}_{01}, & W^{01}_{11}, & W^{21}_{11}, \\ W^{01}_{00}, & W^{00}_{01}, & W^{12}_{01}, & W^{22}_{11}, & W^{12}_{11}, \end{matrix}$$

respectively, while each $\mathbf{p}'(a)$ has the effect of doubling the indices throughout the set.

If the projection refers to the node $(. ; . . 1 - \epsilon^a)$, then it is found that the corresponding index matrix is

$$\begin{bmatrix} 2 & \cdot & \cdot & a \\ \cdot & 2 & \cdot & \cdot \\ \cdot & 2a & 1 & 1 \\ \cdot & \cdot & \cdot & 1 \end{bmatrix},$$

a result used later in § 5. Eight-dimensional forms and the corresponding reduced form in [4], together with the Clifford set arising from the basic Clifford set under the appropriate transformations, are quite easily established in a manner similar to that just stipulated.

Perhaps it is worth remarking that for any matrix $p(t)$ for which we have $\{p(t)\}^2 = 9\,I$, each column of $p(t) - 3\,I$ yields the co-ordinates of the node P_t, whereas each row of $p(t) - 3\,I$ yields the prime co-ordinates of the Jordan prime of P_t. In the case of the matrices $p(a)$, the appropriate matrix whose columns and rows give rise to the node P_a and its Jordan prime respectively, is $p(a) - I$, since $\{p(a)\}^2 = I$.

Besides the 45 involutions (harmonic inversions) just enumerated, there are 270 other involutions associated with **B**. These are analysed in § 5. The aim of this paper is, specifically, to find matrix forms for these 45 + 270 = 315 involutions, to discover their symplectic forms and their augmented forms as matrices of size 9 (also the effect of these on the basic Clifford set), and to relate the invariant spaces of these (collineation) matrices to the invariant configuration **C** we know to exist in [8]. Part of this objective has been achieved in the above section.

3. Invariant spaces of the 45 harmonic inversions. Next, we examine the invariant spaces in [8] generated by the 45 harmonic inversions. For the harmonic inversion $p(t)$, the invariant solid Δ_{tuvw}, or simply Δ_t when there is no confusion, is found to be determined by the 4 points:

$$\text{(i)} \qquad \begin{array}{c} (\,1 \quad \epsilon^t \quad \epsilon^t \quad . \quad . \quad . \quad . \quad . \quad .\,) \\ (\,1 \quad . \quad . \quad \epsilon^u \quad . \quad . \quad \epsilon^u \quad . \quad .\,) \\ (\,1 \quad . \quad . \quad . \quad \epsilon^v \quad . \quad . \quad . \quad \epsilon^v\,) \\ (\,1 \quad . \quad . \quad . \quad . \quad \epsilon^w \quad . \quad \epsilon^w \quad .\,) \end{array}$$

with $t, u, v, w = 0, 1, 2$ and $t + u + v + w \equiv 0 \pmod 3$. Each of these 4 points remains unchanged under the given collineation. Altogether, nine solids Δ pass through each of the points of (i).

Similarly, the invariant [4], Γ_{tuvw}, or Γ_t, is uniquely determined by four primes whose prime co-ordinates are like the set (i), except that the indices are everywhere doubled. By their nature, $p^*(t)$ and $p'^*(t)$ effect the same invariance of spaces. Nine Γ lie in each prime.

Clearly, all the 26 solids Δ_t lie in the [4] Π since, by **(11)** the 12 points in (i) are among the 40 points of Π invariant under JW. Dually, all the 27 Γ_t pass through the solid Σ (corresponding to the Π) determined by the four points $A_{ij} - A_{2i2j}$ $(i, j = 0, 1, 2)$. As we know, Σ lies on L so that each Γ_t cuts L in the same solid Σ.

Furthermore, since the [4]'s Π and a particular Γ_t lie in [8], they must normally meet in a point. Now, by (ii), Π may equally well be determined by the five points A, B_i $(i = 1, 2, 3, 4)$ from which we derive the point $(-2\ \epsilon^t\ \epsilon^t\ \epsilon^u\ \epsilon^v\ \epsilon^w\ \epsilon^u\ \epsilon^w\ \epsilon^v)$ which, from our remarks about the primes determining

Γ_t, obviously lies in the Γ. Of course, this common point is merely the node P_t with which the harmonic inversion is associated. It does not belong to the configuration **C** in [8]. Thus, we have shown that the 27 Γ_t constitute a family of [4]'s through Σ.

Dually, the solid Δ_t, which lies entirely in Π, joins Σ to a prime. In prime co-ordinates, this is $(-2\ \epsilon^t\ \epsilon^t\ \epsilon^u\ \epsilon^v\ \epsilon^w\ \epsilon^u\ \epsilon^w\ \epsilon^v)$ and must not be confused with the prime $(-1\ \epsilon^t\ \epsilon^t\ \epsilon^u\ \epsilon^v\ \epsilon^w\ \epsilon^u\ \epsilon^w\ \epsilon^v)$ whose [4] section by Π produces the Jordan (polar) prime of the node P_t. (Actually, these two primes in [8] meet in a secundum through which the simplex prime $X_{00} = 0$ passes.)

Systematising these results, we have, for the 27 nodes P_t:

each Γ through Σ meets Π in a node;
each Δ in Π joins Σ to a prime.

Irregularities occur in the case of the remaining 18 nodes P_a.

Taking our standard P_a, and considering both $p^*(a)$ and the associated matrix $p'^*(a) = Jp^*(a)$ which produce identical invariant spaces, we find that the [4] Γ_a is determined by the four primes (in prime co-ordinates)

$$
\begin{array}{l}
(\ .\quad 1\quad .\quad \epsilon^{2a}\quad .\quad .\quad .\quad .\quad .\ )\\
(\ .\quad .\quad 1\quad .\quad .\quad .\quad \epsilon^{2a}\quad .\quad .\ )\\
(\ .\quad .\quad .\quad .\quad 1\quad .\quad .\quad .\quad 1\ )\\
(\ .\quad .\quad .\quad .\quad .\quad 1\quad .\quad 1\quad .\ ).
\end{array}
$$

Alternatively, Γ_a is determined by the five points

$$
\begin{array}{rl}
A_{00} \equiv X: & (\ 1\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\ )\\
Y: & (\ .\quad 1\quad .\quad -\epsilon^a\quad .\quad .\quad .\quad .\quad .\ )\\
Z: & (\ .\quad .\quad 1\quad .\quad .\quad .\quad -\epsilon^a\quad .\quad .\ )\\
W: & (\ .\quad .\quad .\quad .\quad 1\quad .\quad .\quad .\quad -1\ )\\
T: & (\ .\quad .\quad .\quad .\quad .\quad 1\quad .\quad -1\quad .\ ),
\end{array}
$$

none of which belongs to the configuration **C**, except X. Points W, T and $Y-Z$ are contained in Σ, while points X and $Y+Z$ belong to Π. Of course, $Y+Z$ is merely the [8] form of the node P_a.

Elements determining Δ_a are either

four points corresponding to the four defining primes of Γ_a, or
five primes corresponding to the five defining points of Γ_a,

with the symbol a replaced everywhere by $2a$ in both cases. None of the four defining points of Δ_a belongs to **C**.

So, for the 18 P_a we find that each Γ_a meets Σ in a plane (not a Jacobian plane) and Π in a line (joining P_a to A_{00}), that is, each Γ_a has three points in Σ and two points in Π, of which A_{00} belongs to **C**.

Similarly, by duality, each Δ_a lies in three primes of Π and two primes of Σ, that is, each Δ_a is the intersection of *a* [5] through Π and *a* [6] (secundum)

through Σ, this secundum being the intersection of the prime dual of P_a and the simplex prime $x_{00} = 0$. Points of **C** are alien to Δ_a though the solid may be shown to possess three points of Σ and one of Π.

With regard to the matrix $p^*(a)$, it is worth noting that there are four possible arrangements of the four units, in pairs, all of which yield a matrix which leaves invariant the node and its corresponding Jordan prime, that is, they are involutory projections. However, as similarity matrices operating on a Clifford matrix W, they do not produce another W (except in the case of the $p^*(a)$ chosen). That is, the other three possibilities are involutions which are not members of the Clifford group CT. Likewise for $p'^*(a)$. Such involutions do not belong to a group since the group multiplication property is absent.

4. Nodes in Jordan primes. Subsequently, it will be found useful to have the forms of the twelve points in a Jordan prime at out disposal. Now the polar (Jordan) prime of the node P_t has prime co-ordinates $(-1\ \epsilon^{2t}\ \epsilon^{2u}\ \epsilon^{2v}\ \epsilon^{2w})$ in [4]. Suppose the node $P_{t'}$ lies in this prime. To satisfy the incidence relation, we must have

$$\text{(ii)} \qquad 2 + \epsilon^{2t+t'} + \epsilon^{2u+u'} + \epsilon^{2v+v'} + \epsilon^{2w+w'} = 0,$$

with the usual restrictions on the variables. That is, $2t + t'$, $2u + u'$, $2v + v'$, $2w + w'$, $\neq 0$. Therefore, $t' = t + 1,\ t + 2$, and similarly for the other variables. Consequently, six nodes are given by

$$\begin{array}{ccccc} -2 & \epsilon^{t+1} & \begin{cases}\epsilon^{u+1}\\ \epsilon^{u+2}\end{cases} & \begin{array}{c}\epsilon^{v+2}\\ \begin{cases}\epsilon^{v+1}\\ \epsilon^{v+2}\end{cases}\end{array} & \begin{array}{c}\epsilon^{w+2}\\ \epsilon^{w+2}\\ \epsilon^{w+1}\end{array}\end{array}$$

and

$$\begin{array}{ccccc} -2 & \epsilon^{t+2} & \begin{cases}\epsilon^{u+2}\\ \epsilon^{u+1}\end{cases} & \begin{array}{c}\epsilon^{v+1}\\ \begin{cases}\epsilon^{v+1}\\ \epsilon^{v+2}\end{cases}\end{array} & \begin{array}{c}\epsilon^{w+1}\\ \epsilon^{w+2}\\ \epsilon^{w+1}\end{array}\end{array}$$

so that in the Jordan prime of a node P_t we have just six nodes of Type I and, therefore, six of Type II.

Regarding nodes P_a, we observe that the corresponding Jordan prime has prime co-ordinates $(\cdot\ 1\ -\epsilon^{2a}\ .\ .)$ in [4], so that only three nodes of Type II are contained therein, namely, the nodes $(\cdot\ \cdot\ \cdot\ 1\ -\epsilon^{a})$. Accordingly, there must be nine nodes of Type I in it.

5. Involutions concerning Jordan pentahedra. Apart from the 45 involutions which leave invariant a node and its Jordan prime and interchange the nodes on each κ-line through the given node, there is another set of involutory operations for which we can find matrix forms. Following Baker **(1)**, Todd **(12)** has shown that such operations of period 2 form a conjugate set, and are the products of pairs of projections.

Now, it is known **(12)** that the twelve nodes of a Jordan prime fall into three sets of 4, forming a triad of desmic tetrahedra, any two of which are in perspective from each vertex of the third. Such a tetrahedron, together with the pole of the Jordan prime, forms a Jordan ***pentahedron***. Otherwise stated, a Jordan pentahedron is a symmetrical set of five nodes such that the polar prime of any one contains the other four, and the join of any two nodes is an ***e-line*** and the plane containing any three nodes is an ***f-plane***, that is, a plane containing three nodes whose joins, in pairs, are e-lines. (By an e-line we mean a line joining a node A to a node B in the Jordan prime of A, and containing no other point.) There are 27 such Jordan pentahedra in **B** (three through a given node), as well as 270 f-planes, 270 e-lines, and 240 κ-lines.

Suppose AB is an e-line. Then

$$p(A)p(B) = p(B)p(A)$$

so that from each e-line there arises one commutative operation $p(A)p(B)$. Todd remarks that $p(A)$ and $p(B)$ generate an Abelian group of order 4, direct product of the cyclic groups generated by the projections $p(A)p(B)$. Further, $p(A)p(B)$ is an axial homology having for invariant spaces the e-line AB and the opposite plane (f-plane) CDE of the Jordan pentahedron having AB for an edge. Consequently, $p(A)p(B)$ leaves invariant the nodes A, B, C, D, E and just these.

Here we may interpolate the result, stated in Todd, that if AB is a κ-line with C the third point on it, then

$$p(A)p(B) = p(B)p(C) = p(C)p(A),$$

so that, under these circumstances, $p(A)p(B)$ has period 3. In addition, Todd remarks that, if $ABCDE$ is a Jordan pentahedron, then

$$p(A)p(B)p(C)p(D)p(E) = 1.$$

Both these results are easy to verify in terms of our matrices.

Because of the different co-ordinate forms for the nodes P_t and P_a, the matrices for $p(A)\,p(B)$ will have varying forms. Having regard to the comments in § 4 about the type of a node in a Jordan prime, we realise that the 270 involutions may be classified in the following way:

For $p(A)p(B)$ of Type I $\times$ Type I, there are $27 \times 6 = 162$ possibilities,
For $p(A)p(B)$ of Type I $\times$ Type II, there are $27 \times 6 + 18 \times 9 = 324$ possibilities,
For $p(A)p(B)$ of Type II $\times$ Type II, there are $18 \times 3 = 54$ possibilities.

However, since each e-line occurs twice in the classification, on account of the commutativity property, we find that the numbers of involutions are 81,162,27 (totalling 270).

Attention is next focused on the set of 81 involutions. On multiplying together the matrices for the projections of two general nodes of Type I,

Table I

$$
\begin{bmatrix}
0 & -(2t)-(2t') & -(2t)-(2t') & -(2u)-(2u') & -(2v)-(2v') & -(2w)-(2w') & -(2u)-(2u') & -(2w)-(2w') & -(2v)-(2v') \\
-(t)-(t') & t+2t' & 2t+t' & t+t'+w+v' & t+t'+u+w' & t+t'+v+u' & t+t'+v+w' & t+t'+u+v' & t+t'+w+u' \\
-(t)-(t') & 2t+t' & t+2t' & t+t'+v+w' & t+t'+w+u' & t+t'+u+v' & t+t'+w+v' & t+t'+v+u' & t+t'+u+w' \\
-(u)-(u') & u+u'+w+v' & u+u'+v+w' & u+2u' & u+u'+t+w' & u+u'+t+v' & 2u+u' & u+u'+v+t' & u+u'+w+t' \\
-(v)-(v') & v+v'+u+w' & v+v'+w+u' & v+v'+t+w' & v+2v' & v+v'+t+u' & v+v'+w+t' & v+v'+u+t' & 2v+v' \\
-(w)-(w') & w+w'+v+u' & w+w'+u+v' & w+w'+t+v' & w+w'+u+t' & w+2w' & w+w'+v+t' & 2w+w' & w+w'+t+u' \\
-(u)-(u') & u+u'+v+w' & u+u'+w+v' & 2u+u' & u+u'+w+t' & u+u'+v+t' & u+2u' & u+u'+t+v' & u+u'+t+w' \\
-(w)-(w') & w+w'+u+v' & w+w'+v+u' & w+w'+v+t' & w+w'+t+u' & 2w+w' & w+w'+t+v' & w+2w' & w+w'+u+t' \\
-(v)-(v') & v+v'+w+u' & v+v'+u+w' & v+v'+w+t' & 2v+v' & v+v'+u+t' & v+v'+t+w' & v+v'+t+u' & v+2v'
\end{bmatrix}
$$

Table II

$$
p(A)p(B) = \begin{bmatrix}
0 & -2(2t) - 2(2t') & -2(2u) - 2(2u') & -2(2v) - 2(2v') & -2(2w) - 2(2w') \\
-(t) - (t') & -(0) & (t + 2u) + (t' + 2u') & (t + 2v) + (t' + 2v') & (t + 2w) + (t' + 2w') \\
-(u) - (u') & (u + 2t) + (u' + 2t') & -(0) & (u + 2v) + (u' + 2v') & (u + 2w) + (u' + 2w') \\
-(v) - (v') & (v + 2t) + (v' + 2t') & (v + 2u) + (v' + 2u') & -(0) & (v + 2w) + (v' + 2w') \\
-(w) - (w') & (w + 2t) + (w' + 2t') & (w + 2u) + (w' + 2u') & (w + 2v) + (w' + 2v') & -(0)
\end{bmatrix}
$$

Table III

$$
p(A)p(B) = \begin{bmatrix}
(w + 2v)(w' + 2v') + 2u't + 2s & st' + 2s't & (w + 2v)t' + (w' + 2v')t & (w + 2v)s' + (w' + 2v')s \\
u + 2u' & (w + 2v)(w' + 2v') + 2ut' + 2s' & w + 2v + w' + 2v' & (w + 2v)2u' + (w' + 2v')2u \\
(w' + 2v')2u + (w + 2v)2u' & (w' + 2v')2s + (w + 2v)2s' & (w + 2v)(w' + 2v') + 2ut' + 2s & u's + 2us' \\
2(w + 2v) + 2(w' + 2v') & (w + 2v)t' + (w' + 2v')t & t + 2t' & (w + 2v)(w' + 2v') + 2u't + 2s'
\end{bmatrix}
$$

Table IV

$$
\begin{bmatrix}
0 & 2t & 2t & 2u & 2v & 2w & 2u & 2w & 2v \\
t & 1 & 2 & 2a & 2v + t + 1 & 2w + t + 2 & 2a & 2w + t + 1 & 2v + t + 2 \\
t & 2 & 1 & 2a & 2v + t + 2 & 2w + t + 1 & 2a & 2w + t + 2 & 2v + t + 1 \\
u & a & a & 2 & 2v + u + 2 & 2w + u + 2 & 1 & 2w + u + 1 & 2v + u + 1 \\
v & 2t + v + 1 & 2t + v + 2 & 2u + v + 2 & 1 & 2w + v + 1 & 2u + v + 1 & 2w + v + 2 & 1 \\
w & 2t + w + 2 & 2t + w + 1 & 2u + w + 2 & 2v + w + 1 & 1 & 2u + w + 1 & 1 & 2v + w + 2 \\
u & a & a & 1 & 2v + u + 1 & 2w + u + 1 & 2 & 2w + u + 2 & 2v + u + 2 \\
w & 2t + w + 1 & 2t + w + 2 & 2u + w + 1 & 2v + w + 1 & 1 & 2u + w + 1 & 1 & 2v + 2 + 1 \\
v & 2t + v + 2 & 2t + v + 1 & 2u + v + 1 & 1 & 2w + v + 2 & 2u + v + 2 & 2w + v + 1 & 1
\end{bmatrix}
$$

say, $A \equiv P_t$ and $B \equiv P_{t'}$, we find, on simplifying by means of (ii) where necessary, and using the convention of **(11)**, that the matrix form of $p^*(A)$ $p^*(B)$ is as shown in Table I. Actually, the convention has been slightly extended, to allow us to interpret (say) $-(2t)-(2t')$ as $-\epsilon^{2t}-\epsilon^{2t'}$, that is, only the portion in brackets is a power of ϵ in cases like this.

Interchanging in pairs rows 2 and 3, 4 and 7, 5 and 9, and 6 and 8,that is, operating on this matrix by J, we obtain the companion matrix $\{p^*(A)\ p^*(B)\}' = p^*(B)\ p^*(A)$ which induces the same collineation as the above matrix. Further, $p^*(B)\ p^*(A)$ is the inverse of $p^*(A)\ p^*(B)$. From these facts, we deduce that

$$\{p^*(A)p^*(B)\}^2 = \{p^*(B)p^*(A)\}^2 = J$$

in [8], that is, the operation $p^*(A)\ p^*(B)$ in [8] is not involutory, but has period 4. On account of the relationship $p'^*(A) = Jp^*(A)$, we have

$$p'^*(A)\ p^*(B) = Jp^*(A)\ p^*(B) = p^*(B)\ p^*(A).$$

Making the necessary adjustments for the form of the projection-product in [4], we have, using our convention again, the equation in Table II. As this is symmetrical in dashed and undashed letters, we have

$$p(A)\ p(B) = p(B)\ p(A)$$

as we expect, and, of course, $\{p(A)\ p(B)\}^2 = I$, that is, the operation is involutory.

Finally, the general symplectic matrix is shown in Table III where $s = (w+2v)^2 + 2ut + 2$, $s' = (w'+2v')^2 + 2u't' + 2$, and the arithmetic is reduced mod 3.

Squaring this matrix always gives $2\,I$ so that its period is 4. Calculation of the square is complicated and tedious.

In particular, consider the nodes $A \equiv P_{0000}$ and $B \equiv P_{1122}$. On calculation, the reduced matrix form for $p(A)p(B)$ is found to be

$$\begin{bmatrix} 1 & 2\epsilon & 2\epsilon & 2\epsilon^2 & 2\epsilon^2 \\ \epsilon^2 & -1 & 2 & -\epsilon & -\epsilon \\ \epsilon^2 & 2 & -1 & -\epsilon & -\epsilon \\ \epsilon & -\epsilon^2 & -\epsilon^2 & -1 & 2 \\ \epsilon & -\epsilon^2 & -\epsilon^2 & 2 & -1 \end{bmatrix}$$

which leaves invariant the points A, B of the e-line and the points $C \equiv P_0$, $D \equiv (\ldots 1 - 1)$ and $E \equiv P_{2211}$ of the f-plane in the Jordan pentahedron. Additionally, in its [8] form, $p^*(A)\ p^*(B)$, as a member of CT, transforms the basic Clifford set into the Clifford set

$$W_{20}^{10} \qquad W_{10}^{00}, \qquad W_{22}^{21}, \qquad W_{21}^{20}, \qquad W_{22}^{22}.$$

Correspondingly, the index matrix is

$$\begin{bmatrix} 1 & 2 & \cdot & \cdot \\ 2 & 2 & \cdot & \cdot \\ \cdot & \cdot & 1 & 2 \\ \cdot & \cdot & 2 & 2 \end{bmatrix}$$

of which the square is $2\,I$.

Next, we examine the invariant spaces associated with the operation $p^*(A)\,p^*(B)$, whose period is 4. Direct calculation shows that the point

$$(1, -(\epsilon^{t} + \epsilon^{t'}), -(\epsilon^{t} + \epsilon^{t'})\ldots)$$

is transformed into the point whose first three co-ordinates are $3\ldots$, and whose succeeding three co-ordinates are

$$\begin{aligned} &-\{2(\epsilon^{u} + \epsilon^{u'}) + \epsilon^{u'+2t'+t} + \epsilon^{u+2t+t'}\} \\ &-\{2(\epsilon^{v} + \epsilon^{v'}) + \epsilon^{v'+2t'+t} + \epsilon^{v+2t+t'}\} \\ &-\{2(\epsilon^{w} + \epsilon^{w'}) + \epsilon^{w'+2t'+t} + \epsilon^{w+2t+t'}\} \end{aligned}$$

with corresponding co-ordinates appropriately situated in the last three positions. Note that the co-ordinates are symmetrical in dashed and undashed symbols so that $p^*(A)\,p^*(B)$ and $p^*(B)\,p^*(A)$ produce the same invariant solid. Similar forms apply for the transformations of the u-, v-, and w-co-ordinates. Only one of the three pairs of co-ordinates in positions 4 and 7, 5 and 9, and 6 and 8, is non-zero, a fact we now proceed to establish.

Firstly, suppose that all three pairs are simultaneously non-zero. Then

$$\begin{aligned} u' + 2t' + t + u + 2t + t' &= u + u' \\ v' + 2t' + t + v + 2t + t' &= v + v' \\ w' + 2t' + t + w + 2t + t' &= w + w' \end{aligned}$$

which imply (say)

(a) $$u' + 2t' + t = u \text{ or } u'$$

(b) $$v' + 2t' + t = v \text{ or } v'$$

(c) $$w' + 2t' + t = w \text{ or } w'$$

(Calculations hereunder are taken modulo 3.) Let $u' + 2t' + t = u'$. Therefore, $2t' + t = 0$ so that $t' = t$ which is clearly impossible since one point lies in the Jordan prime of the other.

Consequently, take $u' + 2t' + t = u$ with similar selections in (b) and (c). Combining these results, we have $2u + u' = 2v + v' = 2w + w' = 2t + t'$ which is obviously invalid from the incidence relation (ii).

Assume next that all three pairs vanish together. Then

$$\begin{aligned} u' + 2t' + t &= u + 2t + t' = 2(u+u') \\ v' + 2t' + t &= v + 2t + t' = 2(v + v') \\ w' + 2t' + t &= w + 2t + t' = 2(w + w') \end{aligned}$$

whence

$$2u + u' = 2v + v' = 2w + w' = t + 2t'$$

which, by virtue of (ii), is essentially inadmissible.

Accordingly, the three pairs cannot be either simultaneously zero or simultaneously non-zero. Suppose, then, that the u-co-ordinates vanish but not the v-co-ordinates. This assumption involves $2u + u' = t + 2t' = 2(2t + t')$ (as above) and $v' + 2t' + t = v$, that is, $v + 2v' = t + 2t'$. Since, from (ii), two powers of ϵ are twice the other two, we must have

$$\begin{aligned} 2w + w' &= 2(2v + v') \\ &= t + 2t' \end{aligned}$$

which means that the w-co-ordinate also vanishes. Analogous reasoning for the vanishing of the v-co-ordinates or of the w-co-ordinates obviously applies. Of the three pairs of co-ordinates, therefore, only one is non-zero.

To illustrate the theory, choose the nodes P_{0120} and P_{1002}. We find that $p(0120)\,p(1002)$ leaves invariant the solid Δ_{2211}. However, this projection-product differs from the projection $p(2211)$, which leaves Δ_{2211} pointwise invariant, in that it effects the substitution $utwv$ on the co-ordinates $t\,u\,v\,w$ as specified in (i). The same solid Δ_{2211} may be shown to arise invariantly from the projection-products $p(0120)\,p(1002)$ and $p(0102)\,p(1020)$ with substitutions $v\,w\,t\,u$ and $w\,v\,u\,t$ respectively of $t\,u\,v\,w$. To see this in (say) the first case, we take the vanishing of the u- and w-co-ordinates with the equation

$$2(\epsilon^{v} + \epsilon^{v'}) + \epsilon^{v'+2t'+t} + \epsilon^{t+2t+t'} = -\,3\epsilon \qquad (t = 0, t' = 1)$$

and solve these equations.

In the manner indicated above, it is verifiable that the 81 invariant solids formed from the projection-products $p(t)\,p(t')$ are merely the 27 solids Δ, each occurring three times. Generally, to the projection $p(T)$ correspond three projection-products $p(t)\,p(2t + 2T)$. This is seen by letting $t' = t + \alpha$ in the projection-product corresponding to the projection $p(T)$. Comparing the t-co-ordinates in the two matrices, we have

$$\begin{aligned} \epsilon^{2T} &= -\,(\epsilon^{2t} + \epsilon^{2t'}) \qquad (t' = t + \alpha) \\ &= \epsilon^{2t+\alpha} \end{aligned}$$

so that $\alpha = 2T + t$. But $\alpha = t' + 2t$, so that $t' + 2t = 2T + t$ whence $t' = 2t + 2T$.

Similarly for the U-, V-, and W-co-ordinates. There does not appear to be any general matrix connection between $p(T)$ and $p(t)p(2t + 2T)$, though in particular cases a simple relation may occur. For instance, $p(2211) = Kp(0000)p(1122)$ where K corresponds to the substitution (174 285 396) and so $K^2 = J$.

Likewise, the 81 [4]'s Γ_t are merely the original 27, each occurring three times.

Turning now to projection-products of Type I × Type II, we find, on using the standard projection $p(a)$ of Type II, that $p(t)p(a)$ yields the matrix in Table IV. When squared, this matrix gives J, showing that it is of period 4. Operating on this matrix by J, we obtain the companion matrix which produces the same invariance of spaces in [8]. Also

$$p^*(a)p^*(t) = J\, p^*(t)p^*(a) = p'^*(t)p^*(a).$$

In [4], the above matrix becomes

$$p(t)p(a) = \begin{bmatrix} 0 & 2(2t) & 2(2u) & 2(2v) & 2(2w) \\ t & -(0) & 2(2a) & -(2v+t) & -(2w+t) \\ u & 2(a) & -(0) & -(2v+u) & -(2w+u) \\ v & -(2t+v) & -(2u+v) & 2(0) & -(2w+v) \\ w & -(2t+w) & -(2u+w) & -(2v+w) & 2(0) \end{bmatrix} = p(a)p(t)$$

of which the square is I.

For the projection-product to refer to an e-line, the incidence relation must be

(iii) $$\epsilon^t = \epsilon^{u+2a}, \quad \text{that is,} \quad t = u + 2a$$

a fact used in the matrix calculation above and again in the determination of the invariant solid hereunder.

It is easy to verify that, as a collineation matrix, $p(t)p(a)$ has the effect of interchanging the t- and u- co-ordinates of the solid Δ, whilst leaving unaltered the v- and w- co-ordinates, that is, of performing the substitution *utvw* on *tuvw*. Clearly, the same Δ arises from the substitutions *vutw*, *wuvt*, *tvuw*, *twvu*, and *tuwv*. Hence, each Δ is repeated six times, thus absorbing the $27 \times 6 = 162$ matrices of this kind. It is noted that the substitution *utvw* corresponds to the node having only t- and u- co-ordinates non-zero, *viz.*, the node P_{ϵ}. Likewise for the other possibilities.

Similarly, each of the 27 [4]'s Γ occurs six times as an invariant space.

Writing $M = w + 2v$, we have, on multiplying, that

$$\mathbf{p}(a)\mathbf{p}(t) = \begin{bmatrix} t & at+s & M & 2aM \\ 1 & 2t & \cdot & M \\ M & aM & 2u & au+2s \\ \cdot & M & 2 & u \end{bmatrix}.$$

Squaring, we get $\{\mathbf{p}(a)\mathbf{p}(t)\}^2 = 2(M^2+1)\,\mathbf{I}$, where $M^2 = 0,1$. Thus, $p(a)p(t)$ is involutory if $M^2 = 1$, that is, if $M = 1, 2$, but is of period 4 if $M = 0$, that is, if $w = v$.

Illustrating the above in the case A and C (given earlier), we have

$$p(A)p(C) = \begin{bmatrix} 1 & 2 & 2 & 2 & 2 \\ 1 & -1 & 2 & -1 & -1 \\ 1 & 2 & -1 & -1 & -1 \\ 1 & -1 & -1 & 2 & -1 \\ 1 & -1 & -1 & -1 & 2 \end{bmatrix}$$

which leaves invariant the nodes A, C of the e-line and the points B, D, E of the f-plane in the Jordan pentahedron. As a (non-involutory) matrix of CT, $p^*(A)\, p^*(C)$ may be shown to transform the basic Clifford set into the Clifford set

$$W^{00}_{10}, \quad W^{20}_{10}, \quad W^{20}_{02}, \quad W^{21}_{02}, \quad W^{21}_{00}$$

and the index matrix performing this operation is

$$\begin{bmatrix} . & 2 & . & . \\ 1 & . & . & . \\ . & . & . & 1 \\ . & . & 2 & . \end{bmatrix}$$

of period 4. Regarded as a collineation matrix in [8], $p^*(A)\, p^*(C)$ is found to leave the solid Δ_{0000} invariant. Incidentally, $p(0000) = \lambda p^*(A) p^*(C)$ where λ corresponds to the substitution (174 396 285) and thus has order 2.

Epitomizing, we see that the effect of the $81 + 162 = 243$ projection-products associated with a projection of Type I, as collineation matrices, is to reproduce each of the 27 solids Δ, and each of the [4]'s Γ in the $3 + 6 = 9$ ways indicated. (These 9 substitutions do not form a group.) So the 243 projection-products make no effective contribution to the sets of invariant spaces.

Finally, consider the last category of projection-products. There are 3 sets of 9 e-lines constituting the 27 e-lines, to which these refer, namely, the e-lines joining

$$\begin{array}{llllll} \text{points } (. & 1 & -\epsilon^a & . & . &) \text{ to points } (. \quad . \quad . \quad 1 \quad -\epsilon^b) \\ \text{points } (. & 1 & . & -\epsilon^c & . &) \text{ to points } (. \quad . \quad 1 \quad . \quad -\epsilon^d) \\ \text{points } (. & 1 & . & . & -\epsilon^e &) \text{ to points } (. \quad . \quad 1 \quad -\epsilon^f \quad . \;), \end{array}$$

$$a \ldots, f = 0, 1, 2.$$

We find that $p^*(a)\, p^*(b)$ has only one non-zero element in each row and column, these being unity in position $\alpha_{00}{}^{00}$, ϵ^{2a} in positions $\alpha_{10}{}^{01}$ and $\alpha_{20}{}^{02}$, ϵ^a in $\alpha_{01}{}^{10}$ and $\alpha_{02}{}^{20}$, ϵ^b in $\alpha_{11}{}^{12}$ and $\alpha_{22}{}^{21}$, and ϵ^{2b} in $\alpha_{21}{}^{11}$ and $\alpha_{12}{}^{22}$. Also,

$$p^*(a)\, p^*(b) = Jp^*(b)\, p^*(a).$$

The reduced form in [4] is

$$p(a)p(b) = \begin{bmatrix} 1 & . & . & . & . \\ . & . & 2a & . & . \\ . & a & . & . & . \\ . & . & . & . & 2b \\ . & . & . & b & . \end{bmatrix},$$

of which the square is $2\,I$ so that $p(a)\,p(b)$ has period 4. Using the symplectic form of $p(b)$ given in § 2, we have, on multiplication,

$$p(a)p(b) = \begin{bmatrix} . & 2b & 1 & 2a \\ . & . & . & 1 \\ 2 & 2a & . & b \\ . & 2 & . & . \end{bmatrix}$$

whose period is 4.

In particular, if $a = 2$, $b = 0$, we may readily verify that $p^*(2)\,p^*(0)$ converts the basic Clifford set into the Clifford set

$$W^{02}_{00}, \qquad W^{00}_{02}, \qquad W^{11}_{02}, \qquad W^{21}_{12}, \qquad W^{11}_{12},$$

and the symplectic matrix performing this function is

$$\begin{bmatrix} . & . & 1 & 1 \\ . & . & . & 1 \\ 2 & 1 & . & . \\ . & 2 & . & . \end{bmatrix}.$$

In [4], $p(2)\,p(0)$ leaves invariant the points D (used earlier) and $F \equiv P_2$ of an e-line and also the points $G \equiv P_{2100}$, $H \equiv P_{1011}$, $I \equiv P_{0222}$ of the f-plane in the Jordan pentahedron.

Under the collineation $p^*(a)\,p^*(b)$ in [8], the solid Δ_{ab} determined by the 4 points

$$\begin{matrix} (& . & 1 & 1 & . & 1 & . & . & . & 1 &) \\ (& . & 1 & 1 & \epsilon^a & . & . & \epsilon^a & . & . &) \\ (& . & . & . & \epsilon^a & . & \epsilon^b & \epsilon^a & \epsilon^b & . &) \\ (& . & . & . & . & 1 & \epsilon^b & . & \epsilon^b & 1 &) \end{matrix}$$

has the second and fourth points left invariant, while the first and third are interchanged. (None of these points belongs to the configuration **C**).

Observation reveals that, as a, b vary within their limits, the 10 points determining the 9 Δ_{ab} lie wholly in Π. Moreover, consideration of the solids Δ_{cd}, Δ_{ef} shows that, in addition to these 10 points, two other points, *viz.*,

$$(.\ 1\ 1\ .\ \epsilon^g\ .\ .\ .\ \epsilon^g) \qquad (g = 1, 2)$$

will arise from the collineations $p^*(c)\,p^*(d)$ and $p^*(e)\,p^*(f)$. Just these 12 points of Π, but not in **C**, thus suffice to define the 27 solids Δ_{ab}, Δ_{cd}, Δ_{ef}.

(Compare these with the twelve points of Π in **C**—the set of points (ii)—which suffice to determine the 27 Δ_t.) But these twelve points of Π also manifestly lie in the simplex prime $x_{00} = 0$ through Σ. Therefore, all 27 Δ_{ab}, Δ_{cd}, Δ_{ef} coalesce into one solid, *viz.*, the solid of intersection of Π and $x_{00} = 0$. I find this an unexpected, but satisfying, simplification. Knowing, however, the perverse forms of matrix operators like $p(a)$ and remembering the importance of our basic spaces, perhaps we should not be too surprised at such eccentricities of behaviour.

Dually, the 27 Γ_{ab}, Γ_{cd}, Γ_{ef} are united as one [4], joining A_{00} to Σ.

So the solid, $\Pi \cap x_{00} = 0$, and the [4], $\Sigma + A_{00}$, are the invariant spaces of the 27 Type II $\times$ Type II operations.

A diagram will serve to emphasize the aesthetic simplicity of the invariant spaces we have been investigating.

6. Summary. Explicit matrix forms for the $45+270 = 315$ involutory operations used by Todd in connection with the configuration **B** have now been found. Extensions of these matrices to [8] and to the symplectic forms have been accomplished. As members of CT, the augmented matrices, numbering $315 \times 2 = 630$, are proved to have period 4 in $270 \times 2 = 540$ cases, and to be involutory in the remaining $45 \times 2 = 90$ cases. Symplectic matrices, 315 in all, are generally of period 4, except the set of 45, which are involutory, and those matrices like $p(a)\, p(t)$ for which $w = v$, which are also involutory. Moreover, the invariant spaces of the collineation matrices in [8] have been found and related to the configuration **C**.

Geometrically, the results may be summed up in the following manner. As collineation matrices in [8], the 630 members of CT leave invariant 315 solids Δ and 315 [4]'s Γ. Of these, only 46 Δ and 46 Γ are distinct. Regarding the Δ, we have:

27 Δ_t lie entirely in Π, being determined completely by twelve points of **C**, and each lies in a prime through Σ;

18 Δ_s each of which lies in three primes through Π and in two primes through Σ, and contains no points of **C**;

1 $\Delta_{ab} \equiv \Pi \cap x_{00} = 0$, containing no points of **C**.

And dually there exist:

27 Γ_t containing Σ entirely, each meeting Π in a Burkhardt node;

18 Γ_s each meeting Σ in a plane and Π in a line, such that the simplex vertex A_{00} is a common point of these eighteen lines;

1 $\Gamma_{ab} \equiv \Sigma + A_{00}$.

Intersection relations amongst the various Δ and Γ may be obtained, but this objective is beyond the purpose of this paper. Details of the intersections of the invariant spaces with the locus L and the configuration **B** may safely be left to the reader's curiosity.

The results embodied in the above summary apply to the [4] Π given by $x_{ij} = x_{2i2j}$ $(i, j = 0, 1, 2)$ and its dual space, the solid Σ defined by $A_{ij} - A_{2i2j}$

$(i, j = 0, 1, 2)$. Obviously, they apply equally well in modified form to the other Π- and Σ- spaces whose invariance is preserved by the collineation matrices JW.

It is hoped that, in another paper, discussion of the complete set of involutions of the Clifford group CT will be forthcoming. The set of 81 JW and 90 p^* (54 $p^*(t)$ and 36 $p^*(a)$) clearly do not form a group, as, for instance, the product of two JW matrices yields merely a W. Also, the product of two p^*'s is a matrix of period 4, as we have shown.

7. Finale. Quite recently, some interesting and important articles have appeared relative to the group CS of order 51840 which is known **(11)** to be the factor group CT/CG, and to the group $\frac{1}{2}CS$ of order 25920.

Structural properties of CS have, within the past few years, been investigated by Edge **(7; 8)**. In the former, he shows that this group has a representation by orthogonal matrices, of size 5 and determinant $+1$, over $GF(3)$.

Dieudonné **(4)** has dealt with geometrical properties associated with two groups of order 25920. These groups, the projective groups corresponding to the symplectic and unitary groups $Sp_4(F_3)$ and $U_4^+(F_4)$ respectively (in Dieudonné's notation), have been proved isomorphic by Dickson **(3)**. Elsewhere, Dieudonné **(5)** discusses the automorphisms of these groups.

By **(11)**, the generators **Q** and **D** of $\frac{1}{2}CS$ have periods 5 and 3 with defining relation $(\mathbf{QD})^5 = \mathbf{I}$. Compare these with the generators S and T of Brahana **(2)**, which have periods 5 and 2 respectively, with ST and S^2T of periods 12 and 9 respectively. Frame **(9)** in examining an abstractly identical group, obtains two generators similar to Brahana's generators.

Finally, CS, the group of automorphisms of the 27 lines of a cubic surface is, Frame **(10)**, the subgroup of the group of order $51840 \times 28 = 1451520$ of the automorphisms of the 28 bitangents to a plane quartic curve, which leaves one bitangent fixed.

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GRAPH THEORY AND PROBABILITY

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A well-known theorem of Ramsay **(8; 9)** states that to every n there exists a smallest integer $g(n)$ so that every graph of $g(n)$ vertices contains either a set of n independent points or a complete graph of order n, but there exists a graph of $g(n) - 1$ vertices which does not contain a complete subgraph of n vertices and also does not contain a set of n independent points. (A graph is called complete if every two of its vertices are connected by an edge; a set of points is called independent if no two of its points are connected by an edge.) The determination of $g(n)$ seems a very difficult problem; the best inequalities for $g(n)$ are **(3)**

$$(1)\qquad 2^{\frac{1}{2}n} < g(n) < \binom{2n-2}{n-1}.$$

It is not even known that $g(n)^{1/n}$ tends to a limit. The lower bound in (1) has been obtained by combinatorial and probabilistic arguments without an explicit construction.

In our paper **(5)** with Szekeres $f(k, l)$ is defined as the least integer so that every graph having $f(k, l)$ vertices contains either a complete graph of order k or a set of l independent points ($f(k, k) = g(k)$). Szekeres proved

$$(2)\qquad f(k, l) < \binom{k+l-2}{k-1}.$$

Thus for

$$k = 3, f(3, l) < \binom{l+1}{2}.$$

I recently proved by an explicit construction that $f(3, l) > l^{1+c_1}$ **(4)**. By probabilistic arguments I can prove that for $k > 3$

$$(3)\qquad f(k, l) > l\binom{k+l-2}{k-1}^{c_2},$$

which shows that (2) is not very far from being best possible.

Define now $h(k, l)$ as the least integer so that every graph of $h(k, l)$ vertices contains either a closed circuit of k or fewer lines, or that the graph contains a set of l independent points. Clearly $h(3, l) = f(3, l)$.

By probabilistic arguments we are going to prove that for fixed k and sufficiently large l

$$(4)\qquad h(k, l) > l^{1+1/2k}.$$

Further we shall prove that

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$$(5) \qquad h(2k+1, l) < c_3\, l^{1+1/k},\ h(2k+2, l) < c_3\, l^{1+1/k}.$$

A graph is called r chromatic if its vertices can be coloured by r colours so that no two vertices of the same colour are connected; also its vertices cannot be coloured in this way by $r - 1$ colours. Tutte (**1**, **2**) first showed that for every r there exists an r chromatic graph which contains no triangle and Kelly (**6**) showed that for every r there exists an r chromatic graph which contains no k-gon for $k \leqslant 5$. (Tutte's result was rediscovered several times, for instance, by Mycielski (**7**). It was asked if such graphs exist for every k.) Now (4) clearly shows that this holds for every k and in fact that there exists a graph of n vertices of chromatic number $> n^{\epsilon}$ which contains no closed circuit of fewer than k edges.

Now we prove (4). Let n be a large number,

$$0 < \epsilon < \frac{1}{k}$$

is arbitrary. Put $m = [n^{1+\epsilon}]$ ($[x]$ denotes the integral part of x, that is, the greatest integer not exceeding x), $p = [n^{1-\eta}]$ where $0 < \eta < \epsilon/2$ is arbitrary. Let $\mathfrak{G}^{(n)}$ be the complete graph of n vertices $x_1, x_2, \ldots, x_n$ and $\mathfrak{G}^{(p)}$ any of its complete subgraphs having p vertices. Clearly we can choose $\mathfrak{G}^{(p)}$ in $\binom{n}{p}$ ways. Let

$$\mathfrak{G}_\alpha^{(n)},\ 1 \leqslant \alpha \leqslant \binom{\binom{n}{2}}{m}$$

be an arbitrary subgraph of $\mathfrak{G}^{(n)}$ having m edges (the number of possible choices of α is clearly as indicated).

First of all we show that for almost all α, $\mathfrak{G}_\alpha^{(n)}$ has the property that it has more than n common edges with every $\mathfrak{G}^{(p)}$. Almost all here means: for all α's except for

$$o\binom{\binom{n}{2}}{m}.$$

Let the vertices of $\mathfrak{G}^{(p)}$ be $x_1, x_2, \ldots, x_p$. The number of graphs $\mathfrak{G}_\alpha^{(n)}$ containing not more than n of the edges (x_i, x_j), $1 \leqslant i < j \leqslant p$ equals by a simple combinatorial reasoning

$$\sum_{l=0}^{n} \binom{\binom{p}{2}}{l}\binom{\binom{n}{2}-\binom{p}{2}}{m-l} < (n+1)\binom{\binom{p}{2}}{n}\binom{\binom{n}{2}-\binom{p}{2}}{m}$$

$$< p^{2n}\binom{\binom{n}{2}-\binom{m}{2}}{m} < \binom{\binom{n}{2}}{m} p^{2n}\left(1 - \frac{\binom{p}{2}}{\binom{n}{2}}\right)^m < \binom{\binom{n}{2}}{m} p^{2n}\left(1 - \frac{p^2}{n^2}\right)^m$$

$$< \binom{\binom{n}{2}}{m} p^{2n} \exp\left(-\frac{mp^2}{n^2}\right).$$

Now the number of possible choices for $\mathfrak{G}^{(p)}$ is

$$\binom{n}{p} < n^p < p^n.$$

Thus the number of α's for which there exists a $\mathfrak{G}^{(p)}$ so that $\mathfrak{G}^{(p)} \cap \mathfrak{G}_\alpha^{(n)}$ has not more than n^ϵedges is less than ($\eta < \epsilon/2$)

$$\binom{\binom{n}{2}}{m} p^{2n} \exp(-n^{1+\epsilon-2\eta}) = o\binom{\binom{n}{2}}{m}$$

as stated.

Unfortunately almost all of these graphs $\mathfrak{G}_\alpha^{(n)}$ contain closed circuits of length not exceeding k (in fact almost all of them contain triangles). But we shall now prove that almost all $\mathfrak{G}_\alpha^{(n)}$ contain fewer than n/k closed circuits of length not exceeding k.

The number of graphs $\mathfrak{G}_\alpha^{(n)}$ which contain a given closed circuit (x_1, x_2), $(x_2, x_3), \ldots, (x_l, x_1)$ clearly equals

$$\binom{\binom{n}{2} - l}{m - l}.$$

The circuit is determined by its vertices and their order—thus there are $n(n-1)\ldots(n-l+1)$ such circuits. Therefore the expected number of closed circuits of length not exceeding k equals

$$\binom{\binom{n}{2}}{m}^{-1} \sum_{l=3}^{k} l! \binom{n}{l} \binom{\binom{n}{2} - l}{m - l} < (1 + o(1)) \sum_{l=3}^{k} n^l \left(\frac{m}{\binom{n}{2}}\right)^l$$

$$< (1 + o(1))\, n^k \frac{(2m)^k}{n^{2k}} = o(n)$$

since $\epsilon < 1/k$. Therefore, by a simple and well-known argument, the number of the α's for which $\mathfrak{G}_\alpha^{(n)}$ contains n/k or more closed paths of length not exceeding k is

$$o\binom{\binom{n}{2}}{m},$$

as stated.

Thus we see that for almost all α $\mathfrak{G}_\alpha^{(n)}$ has the following properties: in every $\mathfrak{G}^{(p)}$ it has more than n edges and the number of its closed circuits having k or fewer edges is less than n/k. Omit from $\mathfrak{G}_\alpha^{(n)}$ all the edges contained in a closed circuit of k or fewer edges. By what has just been said we omit fewer than n edges. Thus we obtain a new graph $\mathfrak{G}_\alpha'^{(n)}$ which by construction does not contain a closed circuit of k or fewer edges. Also clearly $\mathfrak{G}_\alpha'^{(n)} \cap \mathfrak{G}^{(p)}$

is not empty for every $\mathfrak{G}^{(p)}$. Thus the maximum number of independent points in $\mathfrak{G}_\alpha'^{(n)}$ is less than $p = [n^{1-\eta}]$, or

$$h(k, [n^{1-\eta}]) > n$$

which proves (4).

By more complicated arguments one can improve (4) considerably; thus for $k = 3$ I can show that for every $\epsilon > 0$ and sufficiently large l

$$f(3, l) = h(3, l) > l^{2-\epsilon},$$

which by (2) is very close to the right order of magnitude.

At the moment I am unable to replace the above "existence proof" by a direct construction.

By using a little more care I can prove by the above method the following result: there exists a (sufficiently small) constant c_4 so that for every k and l

$$h(k, l) > c_4 \, l^{1+\frac{1}{2k}}. \tag{6}$$

(If $k > c \log l$ (6) is trivial since $h(k, l) > l$.)

From (6) it is easy to deduce that to every r there exists a c_5 so that for $n > n_0(r, c_5)$ there exists an r chromatic graph of n vertices which does not contain a closed circuit of fewer than $[c_5 \log n]$ edges. I am not sure if this result is best possible.

We do not give the details of the proof of (3) since it is simpler than that of (4). For $k = 3$ (3) follows from (4). If $k > 3$, put

$$m = c_6[n^{2-\frac{2}{k-1}}]$$

and denote by $\mathfrak{G}_\alpha^{(n)}$ the "random" graph of m edges. By a simple computation it follows that for sufficiently small c_6, $\mathfrak{G}_\alpha^{(n)}$ does not contain a complete graph of order k for more than

$$0{\cdot}9\binom{\binom{n}{2}}{m}$$

values of α, and that for more than this number of values of α $\mathfrak{G}_\alpha^{(n)}$ does not contain a set of $c_7 n^{2/k-1} \log n$ independent points ($c_7 = c_7(c_6)$ is sufficiently large). Thus

$$f(k, c_7 n^{2/k-1} \log n) > n,$$

which implies (3) by a simple computation.

Now we prove (5). It will clearly suffice to prove the first inequality of (5). We use induction on l. Let there be given a graph $\mathfrak{G}$ having $h(2k + 1, l) - 1$ vertices which does not contain a closed circuit of $2k + 1$ or fewer edges and for which the maximum number of independent points is less than l. If every point of $\mathfrak{G}$ has order at least $[l^{1/k}] + 2$ (the order of a vertex is the number of edges emanating from it) then, starting from an arbitrary point, we reach in k steps at least l points, which must be all distinct since otherwise $\mathfrak{G}$ would

have to contain a closed circuit of at most $2k$ edges. The endpoints thus obtained must be independent, for if two were connected by an edge $\mathfrak{G}$ would contain a closed circuit of $2k + 1$ edges. Thus $\mathfrak{G}$ would have a set of at least l independent points, which is false.

Thus $\mathfrak{G}$ must have a vertex x_1 of order at most $[l^{1/k}] + 1$. Omit the vertex x_1 and all the vertices connected with it. Thus we obtain the graph $\mathfrak{G}'$ and x_1 is not connected with any point of $\mathfrak{G}'$, thus the maximum number of independent points of $\mathfrak{G}'$ is $l - 1$, or $\mathfrak{G}'$ has at most $h(2k + 1, l - 1) - 1$ vertices, hence

$$h(2k + 1, l) < h(2k + 1, l - 1) + [l^{1/k}] + 2$$

which proves (5).

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A CLASS OF FROBENIUS GROUPS

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1. Introduction. If a group contains two subgroups A and B such that every element of the group is either in A or can be represented *uniquely* in the form aba', a, a' in A, $b \neq 1$ in B, we shall call the group an *independent ABA-group*. In this paper we shall investigate the structure of independent ABA-groups of finite order.

A simple example of such a group is the group G of one-dimensional affine transformations over a finite field K. In fact, if we denote by a the transformation $x' = \omega x$, where ω is a primitive element of K, and by b the transformation $x' = -x + 1$, it is easy to see that G is an independent ABA-group with respect to the cyclic subgroups A, B generated by a and b respectively.

Since G admits a faithful representation on m letters (m = number of elements in K) as a transitive permutation group in which no permutation other than the identity leaves two letters fixed, and in which there is at least one permutation leaving exactly one letter fixed, G is an example of a Frobenius group. In Theorem 1 we shall show that this property is characteristic of independent ABA-groups.

In a Frobenius group on m letters, the set of elements whose order divides m forms a normal subgroup, called the *regular subgroup*. In our example, the regular subgroup M of G consists of the set of translations, and hence is an Abelian group of order $m = p^n$ and of type $(p, p, \ldots, p)$. Our main object will be to give a proof (Theorem 5) that the regular subgroup of an independent ABA-group is always an Abelian group of type $(p, p, \ldots, p)$. We shall call such an Abelian group an *elementary* Abelian group. Throughout the paper all groups will be assumed to be of finite order.

2. Independent ABA-groups as Frobenius groups.

THEOREM 1. *If G is an independent ABA-group, then G is a Frobenius group. If A has order h, B has order k, and the regular subgroup M of G has order m, then $m = h(k-1) + 1$.*

Proof. Consider $A \cap xAx^{-1}$ for x in G, and suppose that for some x this intersection contains an element $a \neq 1$. If x is not in A, then by definition of

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G, $x = a'ba''$, a', a'' in A, $b \neq 1$ in B; and consequently $(a'ba'')a_1(a'ba'')^{-1} = a$ for some a_1 in A. It follows that $b\bar{a}_1 = \bar{a}b$, where $\bar{a}_1 = a''a_1a''^{-1}$ and $\bar{a} = a'^{-1}aa'$, which contradicts the fact that G is an independent ABA-group.

Hence $A \cap xAx^{-1} \neq 1$ implies x is in A; thus the normalizer of A in G is A itself and the intersection of A with any of its conjugates consists only of the identity element of G. It is well known that these conditions imply that G is a Frobenius group, and furthermore, if M is the regular subgroup of G, that $G = AM$ (**5**, 144).

Thus the order of G is hm. On the other hand, as an independent ABA-group, the order of G is easily computed to be $h^2(k - 1) + h$, whence the equality $m = h(k - 1) + 1$ follows at once.

3. A class of Frobenius groups. Let $G = AM$ be a Frobenius group, its regular subgroup M having order m, and A of order h. Since the automorphism of M induced by conjugation by an element of $A(\neq 1)$ leaves only the identity element of M fixed, it follows that $h|m - 1$, and hence the quantity $k = 1 + (m - 1)/h$ is an integer. In an independent ABA-group this integer k is, by Theorem 1, the order of the subgroup B, and hence k divides the order hm of G.

In this section we shall completely determine the structure of the regular subgroup of a Frobenius group in which the integer k has this additional property.

THEOREM 2. *Let $G = AM$ be a Frobenius group, M its regular subgroup, of order m, A of order h, and set $1 + (m - 1)/h = k$. Then if $k|hm$, M is either a p-group or the direct product of two elementary Abelian groups.*

Proof. Suppose $p|m$, and let S_p be a p-Sylow subgroup of M. If N_p denotes the normalizer of S_p in G, then N_p is itself a Frobenius group, and in fact $N_p = A'N_p'$ where A' is of order h and N_p' is the normalizer of S_p in M (**3**, Lemma 2.5). Thus N_p' is left invariant by the automorphisms of M induced by A'. Since S_p is a characteristic subgroup of N_p', it also is left invariant by these automorphisms. On the other hand, any two subgroups of order h in G are known to be conjugate, so that $A' = xAx^{-1}$ for some x in G. It follows that the p-Sylow subgroup $x^{-1} S_p x$ is left invariant by the automorphisms of M induced by A.

The set of elements H_p of order dividing p which are in the centre of this p-Sylow subgroup themselves form a subgroup of M which is left invariant by A. It is still possible that some proper subgroup of H_p is invariant under the automorphisms induced by A. Let T_p be a minimal such subgroup; T_p is an elementary Abelian group of order p^n, $n > 1$. Moreover,

3.1 $$h|p^n - 1$$

and *a fortiori* $(h, p) = 1$. Since $T_p \subset M$, we must also have

3.2 $$p^n|m.$$

By definition of k, we also have the equality

$$m = h(k - 1) + 1. \tag{3.3}$$

Using 3.1 and 3.2, it follows easily from this relation that

$$k = \frac{p^n - 1}{h} + 1 + \lambda p^n \tag{3.4}$$

for some integer $\lambda \geqslant 0$, and hence that

$$m = p^n(1 + \lambda h). \tag{3.5}$$

Since $k|hm$, we can write $k = k_1k_2$ where $k_1|h$ and $k_2|m$; and hence using 3.3,

$$k_1|h, \quad k_2|h - 1. \tag{3.6}$$

Thus $k = k_1k_2 < h(h - 1) < hp^n$, and consequently

$$\lambda < h. \tag{3.7}$$

Suppose now that M is not a p-group and hence that there is a prime $q \neq p$ dividing m. As above, M contains a minimal elementary Abelian subgroup T_q of order q^r, $r \geqslant 1$, which is invariant under A. Thus

$$h|q^r - 1, \tag{3.8}$$

and as $q^r|m$,

$$q^r|1 + \lambda h. \tag{3.9}$$

It follows from 3.8 that $q^r = 1 + \mu h$ for some $\mu \geqslant 1$, whence $1 + \lambda h = \gamma(1 + \mu h)$ for some $\gamma \geqslant 1$, by 3.9. Thus $\gamma \equiv 1 \pmod{h}$; and hence the assumption $\gamma > 1$ implies $\gamma > h$, whence $1 + \lambda h > 1 + h^2$, contrary to the fact that $\lambda < h$. Hence $\gamma = 1$, $\mu = \lambda$, $q^r = 1 + \lambda h$, and we conclude that

$$m = p^n q^r. \tag{3.10}$$

It follows now from Burnside's well-known theorem that M is solvable, and hence by a theorem of Feit **(3)** and Higman **(4)**, M is in fact nilpotent. Thus M is the direct product of the elementary Abelian groups T_p and T_q, and the theorem is proved.

COROLLARY. *Under the hypothesis of Theorem* 2, *G is solvable if A is solvable.*

Proof. $G/M = A$, and, by the theorem, M is solvable.

The structure of M can, however, be determined much more explicitly:

THEOREM 3. *Under the hypothesis of Theorem* 2, *the regular subgroup M of G is either*

I. *An elementary Abelian group,*

II. *An abelian group of order* 16 *and of type* (4, 4), *with* $h = 3$,

III. *The direct product of two elementary Abelian groups whose orders* p^n *and* q^r *are connected by the equalities*

$$2 + p^{\frac{1}{2}n} = q^r = h + 1.$$

Proof. We preserve the notation of Theorem 2.

Case 1. M is a p-group. If $m = p^t$, we must have $t \geqslant n$, since $T_p \subset M$. If $t = n$, $M = T_p$ and there is nothing to prove. Hence we may assume $t > n$.

For suitable integers μ and s, we have

3.11 $$h = 1 + \mu p^s,$$

where $(\mu, p) = 1$ and $s < n$. Since $h|p^t - 1$ and $h|p^n - 1$, $h|p^{t-n} - 1$, and hence

3.12 $$\mu p^s < p^{t-n}.$$

Furthermore, by definition of k, we have

$$k = \frac{p^t - 1}{1 + \mu p^s} + 1 = p^s \frac{p^{t-s} + \mu}{h},$$

whence

3.13 $$k_1 = \frac{p^{t-s} + \mu}{h}, \qquad k_2 = p^s.$$

It follows at once that

3.14 $$(p^{t-s} + \mu)|(p^n - 1)^2.$$

Now $(p^n - 1)^2 = p^{2n-(t-s)}(p^{t-s} + \mu) - (2p^n + \mu p^{2n-(t-s)} - 1)$, and consequently

3.15 $$(p^{t-s} + \mu)|2p^n + \mu p^{2n-(t-s)} - 1.$$

Thus

3.16 $$\mu p^{2n-(t-s)} \geqslant p^{t-s} - 2p^n + \mu + 1.$$

But now, using 3.12 we have $t - s > n$; combining this inequality with the right-hand side of 3.16, yields

3.17 $$\mu p^{2n-(t-s)} \geqslant p^{t-s-1}$$

except when $p = 2$ and $n = t - s - 1$.

Leaving this exceptional case aside for the moment, we see that 3.17 implies $\mu p^s \geqslant p^{2t-2n-s-1} \geqslant p^{t-n}$ since $t - s - n - 1 \geqslant 0$, and this contradicts 3.12.

We have thus proved that either $t = n$ or $p = 2$ and $t = n + s + 1$. Since $h|p^{t-n} - 1$, we have in the latter case $(1 + \mu 2^s)|2^{s+1} - 1$, whence $\mu = 1$, $s = 1$, $h = 3$. But now 3.6 becomes $\frac{1}{3}(2^{n+1} + 1)|3$, and hence $n = 2$, $t = 4$. Thus M is a group of order 16, while T_2 is of order 4.

Since $h = 3$, M must admit an automorphism of order 3 leaving no elements other than the identity fixed. It can be shown that a group of order 16 having

such an automorphism is either an elementary Abelian group or an Abelian group of type (4, 4).

We have therefore proved that if M has prime-power order, then it is in fact an elementary Abelian group, with the single exception stated in II.

Case 2. M is not a p-group. Then by Theorem 2, M is the direct product of elementary Abelian groups M_p of order p^n and M_q of order q^r. This time we write

3.18 $$h = 1 + \mu p^s q^t, \qquad (\mu, pq) = 1,$$

and as above

3.19 $$\mu p^s q^t < p^n, \qquad \mu p^s q^t < q^r.$$

From the definition of k, we also have

(3.20) $$k_1 = \frac{p^{n-s} q^{r-t} + \mu}{h}, \qquad k_2 = p^s q^t.$$

Furthermore, $(p^{n-s} q^{r-t} + \mu) | (p^n - 1)(q^r - 1)$, and hence, as in Case 1,

3.21 $$\mu p^s q^t > p^{n-s} q^{r-t} - p^n - q^r + \mu + 1.$$

We shall suppose, for definiteness, that $p^n > q^r$, and hence that

$$\mu p^s q^t > p^n \left[\frac{q^{r-t}}{p^s} - 2 \right].$$

In view of 3.19, the quantity in the brackets is less than 1, whence

3.22 $$q^r < 3 p^s q^t.$$

Using 3.19 again, it follows that $\mu \leqslant 2$. However, 3.19 can be strengthened considerably; in fact, it is clear that $2\mu p^s q^t < q^r$ unless $h = q^r - 1$, and $3\mu p^s q^t < q^r$ unless $h = q^r - 1$ or $2h = q^r - 1$. It follows therefore from 3.22 that

3.23 $$\nu(1 + \mu p^s q^t) = q^r - 1,$$

where $\nu = 1$ or 2 if $\mu = 1$, and $\nu = 1$ if $\mu = 2$.

We deduce by inspection that 3.23 has the following five solutions only:

3.24
$$\begin{array}{lll}
(a) & t = 0, \mu = 1, q \neq 2, & \nu = 1 \\
(b) & t = 0, \mu = 2, q = 2, & \nu = 1 \\
(c) & t = 0, \mu = 1, q \neq 3, & \nu = 2 \\
(d) & t = 1, \mu = 1, q = 2, & \nu = 1 \\
(e) & t = 1, \mu = 1, q = 3, & \nu = 2.
\end{array}$$

In particular, it follows from this that

3.25 $$h = 1 + \alpha p^s,$$

where $1 \leqslant \alpha \leqslant 3$.

Since $h|p^n - 1$, we have $p^n - 1 = \gamma(1 + \alpha p^s)$, $\gamma \geq 1$, and hence $\gamma = -1 + \beta p^s$, $\beta \geq 1$. Upon substitution for γ, we obtain

$$\beta\alpha p^{2s} = p^n + (\alpha - \beta)p^s. \tag{3.26}$$

Since $\alpha \leq 3$, the assumption $n < 2s$ implies $\beta = 0$, which is impossible. Thus $n \geq 2s$.

Consider next the case $n = 2s$. The only solution of 3.26 is then easily seen to be $\alpha = 1, \beta = 1$. This implies that we are either in Case 3.24 (a) or 3.24 (c). However, Case 3.24 (c) with $n = 2s$ yields $q^r = 3 + 2p^s$, and hence

$$k_1 = \frac{p^s(3 + 2p^s) + 1}{1 + p^s} = 2p^s + 1.$$

This is impossible since $k_1|h$ and $h = 1 + p^s$.

In Case 3.24(a), on the other hand, we obtain the solution $h = 1 + p^s = q^r - 1$, $k_1 = 1 + p^s$, $k_2 = p^s$, which accounts for the third alternative of the theorem.

We may therefore assume throughout the remainder of the proof that $n > 2s$. Consider first the cases in which $t = 0$. We use 3.23 to replace q^r in 3.21, obtaining

$$(\nu\mu + \mu)p^s \geq (\nu\mu - 1)p^n + (1 + \nu)p^{n-s} + \mu - \nu. \tag{3.27}$$

In each of the three cases in which $t = 0$ this inequality implies that $n \leq 2s$, contradicting our present assumption that $n > 2s$.

Similarly in Case 3.24(d), 3.21 reduces to

$$4p^s \geq p^{n-s}. \tag{3.28}$$

Either $n \leq 2s$ or, since $q = 2$, $p = 3$ and $n = 2s + 1$. But this would require $1 + 2.3^s|3^{2s+1} - 1$, which is impossible.

Finally in Case 3.24(e), 3.21 reduces to

$$9p^s \geq p^n + p^{n-s} - 1. \tag{3.29}$$

Since $q = 3$, it follows that $n \leq 2s$ except when $p = 5$, $s = 0$, $n = 1$ or $p = 2$, $n \leq 2s + 2$. In the first case, $p^n = 5$, $q^r = 9$, contrary to our assumption $p^n > q^r$. The second case requires either $1 + 3.2^s|2^{2s+1} - 1$ or $1 + 3.2^s|2^{2s+2} - 1$, the only solution of which is easily checked to be $s = 1$. But then $2h = 14$, which is not of the form $3^r - 1$. This completes the proof.

Corollary. *If M is an elementary Abelian group, A is a maximal subgroup of G, except when the order of M is 16 and the order of A is 3.*

Proof. In Case 1 of the proof of the theorem, we actually showed that $M = T_p$, except when $p = 2$, T_p is of order 4, and M is of order 16. Since by construction no proper subgroup of T_p is left invariant by A, the equality $M = T_p$ clearly implies that A is a maximal subgroup of G.

4. Independent ABA-groups in which A is of even order. The following theorem gives the complete structure of independent ABA-groups in which A has even order. Its proof does not depend upon Theorems 2 and 3, but only on the fact that such a group is a Frobenius group. This theorem will be used in the next section in the proof of our main result (Theorem 5).

THEOREM 4. *Let G be an independent ABA-group in which the order h of A is even, and let m be the order of the regular subgroup M of G. Then $h = m - 1$, M is an elementary Abelian group, A is isomorphic to the multiplicative group of a nearfield K, and G is isomorphic to the one-dimensional affine group over K.*

Proof. Since h is even, A contains an element a^* of order 2. Let $\sigma_{a^*}(t) = a^{*-1}ta^*$ for all t in M. Then σ_{a^*} is an automorphism of M or order 2 leaving only the identity element fixed. But a group having such an automorphism can easily be shown to be Abelian. (**1**, p. 90).

It follows therefore that

$$\sigma_{a^*}(t\sigma_{a^*}(t)) = \sigma_{a^*}(t)\sigma_{a^*}^2(t) = \sigma_{a^*}(t)t = t\sigma_{a^*}(t).$$

Thus $t\sigma_{a^*}(t)$ is left fixed by σ_{a^*}, and hence equals 1. We conclude that

$$a^*t^{-1} = ta^* \tag{4.1}$$

for all t in M.

Now let $b \in B$, $b \neq 1$. Since $G = AM$, we can write $b = at$, $a \in A$, $t \in M$. If $a = 1$, b is in M, and then 4.1 implies $a^*b^{-1} = ba^*$, contradicting the independence of G.

Thus $a \neq 1$. Suppose, if possible, that $a \neq a^*$. Let a have order d, and put $\sigma_a(t) = a^{-1}ta$. Then

$$b^{d-1} = (at)^{d-1} = a^{d-1}[\sigma_a^{d-2}(t). \ldots . \sigma_a(t)t] = a^{d-1}t',$$

where t', in M, denotes the quantity in brackets. Since M is Abelian, $\sigma_a^{d-1}(t)t'$ is left fixed by σ_a, and hence $\sigma_a^{d-1}(t)t' = 1$. Thus

$$b^{d-1} = a^{d-1}[\sigma_a^{d-1}(t)]^{-1}. \tag{4.2}$$

But now it follows from 4.1 that

$$b^{d-1}a^* = a^{d-1}a^*\sigma_a^{d-1}(t). \tag{4.3}$$

On the other hand, $ba^{-1} = (at)a^{d-1} = \sigma_a^{d-1}(t)$, and consequently

$$b^{d-1}a^* = a^{d-1}a^*ba^{-1}. \tag{4.4}$$

Since $a^* \neq a$, this contradicts the independence of G.

We conclude then that every element of B distinct from the identity is of the form a^*t with t in M. If B contained two such elements $b_1 = a^*t_1$ and $b_2 = a^*t_2$, it would follow that $b = b_1b_2 = t_1^{-1}t_2$ were in $M \in B$, and we have already shown that this leads to a contradiction.

It follows therefore that B has order 2, and hence that $m = h(k-1)+1 = h+1$, thus establishing the first conclusion of the theorem.

But the structure of a Frobenius group of order $(m-1)m$, where m is the order of its regular subgroup M, is well-known (compare **2**, chapters VI, X, XIII): M is an elementary Abelian group, G is isomorphic to the one-dimensional affine group over a near field K of order m, and under this isomorphism, the subgroup A of G is mapped onto the multiplicative group of K.

5. The Structure of independent ABA**-groups.** We are now in a position to establish our main result:

THEOREM 5. *The regular subgroup M of an independent ABA-group G is an elementary Abelian group. Moreover, A is a maximal subgroup of G.*

Proof. By Theorem 3, M is either an elementary Abelian group, an Abelian group of type (4, 4) with $h = 3$, or the direct product of two elementary Abelian groups M_p, M_q of orders p^n, q^r satisfying the relations: $h+1 = 2+p^{\frac{1}{2}n} = q^r$.

That no independent ABA-group of the third type exists may be seen as follows: since $p \neq q$, we must have $p \neq 2$, and hence h is even. But then Theorem 4 implies $h = m-1 = p^n q^r - 1$, contrary to the fact that $h = q^r - 1$.

On the other hand, by the corollary of Theorem 3, if M is an elementary Abelian group, A is a maximal subgroup of G except when M has order 16 and $h = 3$. Thus the theorem will be completely proved if we show that no independent ABA-group exists in which $h = 3$ and M is either an elementary Abelian group or an Abelian group of type (4, 4).

From the relation $h(k-1)+1 = m$ with $h = 3$, $m = 16$, we conclude that k = order of $B = 6$. Since G is a Frobenius group, every element is either in M or conjugate to an element of A. Thus the elements of G are of orders 1, 2, 3 or 4; and hence B is not cyclic. Consequently B is generated by elements b_1, b_2 of orders 2, 3 respectively satisfying the relation

5.1 $$b_1 b_2 b_1^{-1} = b_2^{-1}.$$

Since b_1 is of order 2, it is in M. On the other hand, $b_2 = a^{\epsilon}t$, where t is in M, and $\epsilon = \pm 1$. Thus $b_1 a^{\epsilon} t\, b_1^{-1} = (a^{\epsilon}t)^{-1}$. Since M is normal in G, it follows at once that $a^{2\epsilon}$ is in M, contrary to the fact that $A \cap M = 1$.

From Theorem 5 we can now deduce the following structure theorem for independent ABA-groups:

THEOREM 6. *Let G be an independent ABA-group with A of order h and the regular subgroup M of G of order m. Then:*

I. *If $h = m-1$, A is isomorphic to the multiplicatie group of a nearfield K, and G is isomorphic to the one-dimensional affine group over K. Conversely, the one-dimensional affine group over any finite nearfield is an independent ABA-group satisfying these conditions.*

II. *If $h < m - 1$, A is a metacyclic group of odd order whose generators a_1, a_2 satisfy the relations*

$$a_1^{h_1} = a_2^{h_2} = 1, a_2 a_1 a_2^{-1} = a_1^r, r^{h_2} \equiv 1 \pmod{h_1}, \quad \text{and} \quad ((r-1)h_2, h_1) = 1.$$

In particular, if A is cyclic, G is isomorphic to a subgroup of the one-dimensional affine group over a finite field.

Proof. The proof of I has been given in the last paragraph of Theorem 4.

Conversely, the one-dimensional affine group over a finite nearfield K is easily seen to be an independent ABA-group when A is defined to be the set of transformations $x' = ax$, $a \in K$, $a \neq 0$, and B is the subgroup of order 2 generated by the transformation $x' = -x + 1$.

If $h < m - 1$, A is of odd order by Theorem 4. Since A is isomorphic to a group of automorphisms of M, each of which, except the identity, leaves only the identity element of M fixed, it follows that the Sylow subgroups of A are all cyclic (**1**; **2**; **7**). But then it follows that A is a metacyclic group satisfying the conditions listed in II (**6**, 145).

Finally if A is cyclic, we denote by σ_a the automorphism of M induced by a generator a of A. For convenience, we also regard M as an n-dimensional vector space over the integers modulo p. Since A is maximal in G, no subspace of M is left invariant by A, and hence the elements $t, \sigma_a(t), \ldots, \sigma_a^{n-1}(t)$ are linearly independent over the integers mod p for every $t \neq 0$ in M. For each choice of the integers $c_0, c_1, \ldots, c_{n-1}$ (mod p), not all 0 (mod p), it follows that the mapping

$$t \to \sum_{i=0}^{n-1} c_i \sigma_a^i(t) \tag{5.2}$$

is an automorphism of M leaving only the identity element fixed. In this way we obtain a group of automorphisms A^* of M of order $p^n - 1$, which clearly contains A. It is easy to see that A^* is also cyclic. Hence the Frobenius group $G^* = A^*M$ of order $(p^n - 1)p^n$ is isomorphic to the one-dimensional affine group over $GF(p^n)$. Since $G \subset G^*$, the last statement of the theorem now follows.

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A FURTHER EXTENSION OF CAYLEY'S PARAMETERIZATION

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1. Introduction. In a recent paper **(3)*** the following theorem was proved for real matrices.

THEOREM 1. *If A is a symmetric matrix and Q is a skew-symmetric matrix such that $A + Q$ is non-singular, then*

$$P = (A + Q)^{-1}(A - Q) \tag{1}$$

is a cogredient automorph (c.a.) *of A whose determinant is $+1$ and having the property that A and $I + P$ span the same row space.*

Conversely, if P is a c.a. *of A whose determinant is $+1$ and if P has the property that $I + P$ and A span the same row space, then there exists a skew-symmetric matrix Q such that P is given by equation* (1).

Theorem 1 reduces to the well-known Cayley parameterization in the case where A is non-singular. A similar and somewhat simpler result (Theorem 4) was given for the case when the underlying field is the complex field. It was also shown that the second part of the theorem (in either form) is false when the characteristic of the underlying field is 2. The purpose of this paper is to simplify the proof of Theorem 1 and at the same time, to extend these results to matrices over an arbitrary field of characteristic $\neq 2$.

2. Matrices Over Arbitrary Fields. Let F be a field whose characteristic is not 2 and let $\lambda: a \rightarrow \bar{a}$ be an involutory automorphism on F. We will designate by F_0 the set of elements of F which can be expressed as $a/\bar{a}$ for some $a \in F$. If λ is not the identity mapping, we may also characterize F_0 as the set of elements x for which $x\bar{x} = 1$. If $x \neq -1$, set $a = x + 1$; if $x = -1$, set $a = c - \bar{c}$ where $c \neq \bar{c}$. Then F_0 is a subgroup of the multiplicative group of F.

We define the *conjugate transpose* A^* **(1, 2)** of the matrix $A = [a_{ij}]$, $a_{ij} \in F$ by $A^* = [b_{ij}]$, $b_{ij} = \bar{a}_{ji}$. Then A is *symmetric* if $A^* = A$ and is skew-symmetric if $A^* = -A$.

THEOREM 1'. *If A is a symmetric matrix and Q is a skew-symmetric matrix such that $A + Q$ is non-singular, then*

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$$P = (A + Q)^{-1}(A - Q)$$

is a c.a. *of* A, $|P| \in F_0$ *and* $I + P$ *spans the same row space as* A.

Conversely, if P *is a* c.a. *of* A *such that* $I + P$ *spans the same row space as* A *and if* $|P| \in F_0$, *then there exists a skew-symmetric matrix* Q *such that* P *is given by equation* (1).

Theorem 1 is a special case of Theorem 1′ since F_0 contains just the element $+1$ when F is the real field.

The proof of Theorem 1′ is in three parts. As in the proof of Theorem 1, the first half is immediate. The first part of the proof of the converse consists of repeated simplifications of the forms of A and Q and is analogous to the first part of the proof of the converse in Theorem 1. However, in **(3)** orthogonal transformations $A \rightarrow U'AU$ were used to simplify the forms of all of the matrices involved in the construction of the real skew-symmetric matrix Q (for example, the reduction of A to diagonal form). Since the orthogonal reductions possible in the real field are, in general, not possible in the arbitrary field now under consideration, it is necessary that orthogonality conditions be avoided. Hence, Lemma 1 must be replaced by

LEMMA 1′. *For any non-singular matrix* U, $U^{-1}PU$ *is a* c.a. *of* U^*AU *if and only if* P *is a* c.a. *of* A. *Equation* (1) *holds if and only if*

$$U^{-1}PU = (U^*AU + U^*QU)^{-1}(U^*AU - U^*QU).$$

Moreover, $|P| = |U^{-1}PU|$, *and* $I + P$ *spans the same row space as* A *if and only if* $I + U^{-1}PU$ *spans the same row space as* U^*AU.

Using Lemma 1′, A may be expressed as the direct sum of a non-singular diaongal matrix and a zero matrix **(2**, Theorem 36.2**)**.

The last part of the proof consists of the verification of two conditions (analogous to conditions (6′) and (6″) of **(3)**.)

Condition 1. A non-singular skew-symmetric matrix Z of order $n - 2r + s$ can be constructed. If λ is the identity mapping, it follows from Lemma 2 that $n - 2r + s$ is even and hence a non-singular, skew-symmetric matrix Z exists. If λ is not the identity mapping, then for some $a \in F$, $a \neq \bar{a}$. Hence $(a - \bar{a})/(\bar{a} - a) = -1 \in F_0$ and thus $(a - \bar{a})I$ is non-singular and skew-symmetric.

Condition 2. If B is a c.a. of a non-singular, symmetric, diagonal matrix d, then $I + B$ is a Pr matrix. First we prove

LEMMA. *If the rank of* B *is* r, *then the* r *by* r *matrix* C *formed by the intersection of a set of* r *linearly independent rows of* B *and of* r *linearly independent columns of* B *is non-singular.*

Proof. For convenience, we shall assume that C is in the upper-left-hand corner of B. Let

$$B = \begin{bmatrix} C & D \\ E & F \end{bmatrix}.$$

Since the rows $[CD]$ of B are linearly independent and are r in number, the rows $[EF]$ are linear combinations of them. Thus, there exists a matrix M such that $E = MC$, $F = MD$. Similarly, $D = CN$, $F = EN = MCN$ for some N. Thus,

$$B = \begin{bmatrix} I & O \\ M & I \end{bmatrix} \begin{bmatrix} C & O \\ O & O \end{bmatrix} \begin{bmatrix} I & N \\ O & I \end{bmatrix}$$

and hence C is non-singular.

Proof of Condition 2. Since B is a c.a. of d, $I + B = d^{-1}(I + (B^*)^{-1})d$ and hence, if any set of rows of $I + B$ are linearly independent, the same set of rows of $I + (B^*)^{-1}$ are linearly independent and hence the corresponding set of columns of $I + \bar{B}^{-1} = \bar{B}^{-1}(I + \bar{B})$ are linearly independent, where $\bar{B}$ is the conjugate of B, that is, if $B = (b_{ij})$ then $\bar{B} = (\bar{b}_{ij})$. Consequently, the corresponding set of columns of $I + B$ are linearly independent. By the above lemma, the principal submatrix of $I + B$ determined by these rows is non-singular.

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SOME $\mathfrak{G}$ DIVISION ALGEBRAS

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Introduction. Let K^* be an associative algebra over a field F with identity u, and let $u, e_1, e_2, \ldots,$ be a basis for K^*. Denote by K the linear space, over F, spanned by the e_i, $i = 1, 2, \ldots$. Then for x, y in K, $xy = \alpha u + a$, where $a \in K$. Define $h(x, y) = \alpha$ and $x \cdot y = a$. With respect to the operation $\cdot$ thus defined, K becomes an algebra over F satisfying

$$(x \cdot y) \cdot z - x \cdot (y \cdot z) = h(y, z)x - h(x, y)z. \tag{1}$$

Further, the bilinear form $h(x, y)$ is associative on K. Any algebra, over a field F, which possesses an associative bilinear form $h(x, y)$ and satisfies (1) will be called a $\mathfrak{G}$ algebra. It is not difficult to show that any $\mathfrak{G}$ algebra K can be obtained from a unique associative algebra K^* with identity by the process described above. The algebra K^* will be called the associated associative algebra of K.

A well-known example of a $\mathfrak{G}$ algebra is the Lie algebra of the Euclidean three-dimensional rotation group (obtained from the real quaternions, with a suitable basis, by the process described above). This paper is concerned with the existence of $\mathfrak{G}$ division algebras which are not associative (every associative algebra is obviously a $\mathfrak{G}$ algebra). It will be shown that several non-associative division algebras which appear in the literature are isotopes of $\mathfrak{G}$ division algebras. The observation that a certain three-dimensional division algebra of Dickson **(3)** is an isotope of a $\mathfrak{G}$ division algebra leads to the construction of a new central division algebra of dimension nine. This is the main result.

It is apparent from the process, described above, for obtaining $\mathfrak{G}$ algebras that two non-equivalent $\mathfrak{G}$ algebras may have equivalent associated associative algebras. It can be shown that if K and L are $\mathfrak{G}$ algebras then K^* is equivalent to L^* if and only if there exist a one to one linear mapping T of K onto L and a linear functional f on K such that

$$(xoy)T = f(x)\,(yT) + f(y)\,(xT) + xT * yT$$

holds for all x, y in K, where o, $*$ are the multiplications in K and L respectively. Two algebras related in this way are said to be pseudo-equivalent.

1. A necessary condition. Theorem 1 gives a necessary condition for a $\mathfrak{G}$ division algebra. It seems appropriate to include Theorem 2, since all of the algebras described in §2 satisfy its hypotheses.

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The following lemmas are used in the proof of Theorem 1.

LEMMA 1. *Let A be an associative algebra of (finite) dimension $n > 2$ over a field F. If A contains no proper right ideals then A is a division algebra.*

Proof. Suppose A is not a division algebra. Then there exist in A elements $a \neq 0 \neq b$ such that $ab = 0$. Let $B = \{x \in A \mid ax = 0\}$.

Clearly B is a right ideal and $B \neq 0$, hence $B = A$. Thus $ax = 0$ for all x in A. But then the linear subspace spanned by a is a right ideal of dimension one. This contradiction proves the lemma.

The proof of the following lemma is obtained by a simple computation and is omitted.

LEMMA 2. *Let K be a $\mathfrak{S}$ algebra over a field F with bilinear form $h(x, y)$, and K^* the set of all pairs (ξ, x), $\xi \in F$, $x \in K$. If addition and multiplication are defined for K^* by*

$$(\xi, x) + (\eta, y) = (\xi + \eta, x + y),$$
$$(\xi, x)\cdot(\eta, y) = (\xi\eta + h(x, y), \xi y + \eta x + xy),$$

then K^* is an associative algebra isomorphic to the associated algebra of K.

LEMMA 3. *Let K be a $\mathfrak{S}$ division algebra of dimension $n > 2$ over F and K^* the associated associative algebra. If B^* is a proper right (left) ideal in K^* then the dimension of B^* is either n or 1. Further, K^* contains at most one right (left) ideal of dimension n and at most one of dimension 1.*

Proof. Let $e_1, \ldots, e_n$ be a basis for K. Then, by Lemma 2, $(1, 0)$, $(0, e_1)$, $\ldots$, $(0, e_n)$ is a basis for K^*. If B^* is a proper right ideal then it has a basis (α_1, f_1), (α_2, f_2), $\ldots$, (α_t, f_t), where $t < n$. Suppose all of the $\alpha_i = 0$, then,

$$(0, f_i)(0, a) = (h(f_i, a), f_ia) = \left(0, \sum_{j=1}^{t} \gamma_{ij} f_j\right),$$

for $i = 1, \ldots, t$, and all a in K. Thus the subspace of K spanned by the f_i is a right ideal in K. It follows that $t = n$, and incidentally that K is associative. If $t < n$ then not all of the $\alpha_i = 0$. Without loss of generality suppose that $\alpha_1 = 1$, then $(1, g_1)$, $(0, g_2)$, $\ldots$, $(0, g_t)$ is a basis for B^*, where $g_1 = f_1$, $g_i = f_i - \alpha_i f_1$, $i = 2, \ldots, t$. Since $t < n$ there is an x in K independent of $g_1, \ldots, g_t$. Suppose $t > 1$, and let z be the solution in K of the equation $g_2 z = x$. Clearly $(0, g_2)$ $(0, z) = (h(g_2, z), x)$ is an element of B^*. Thus

$$x = \sum_{i=1}^{t} \mu_i g_i,$$

contrary to the choice of x. Hence $t = 1$, and this completes the first part of the lemma.

If K^* has two right ideals of dimension 1, say B^*_1 and B^*_2 then the sum $B^*_1 + B^*_2$ is a right ideal of dimension 2, which is not possible since $n > 2$.

Further, if K^* has two right ideals of dimension n, say C^*_1 and C^*_2 then since $C^*_1 + C^*_2 = K^*$ it follows that the right ideal $C^*_1 \cap C^*_2$ has dimension $n - 1$. Again this is impossible since $n > 2$. This completes the proof of the lemma.

LEMMA 4. *With K and K^* defined as in Lemma 3 let B^* be a right ideal of dimension n in K^*. If C^* is a proper right ideal in B^* then C^* is a two-sided ideal in K^*.*

Proof. First note that any right ideal in B^* is a right ideal in K^*. Thus, since C^* has dimension $< n$, it follows from Lemma 3 that the dimension of C^* is 1. Let $v^*, w^*_2, \ldots, w_n^*$ be a basis for B^*, where v^* is a basis for C^*. Since $n > 2$ there is a b^* in B^*, $b^* \notin C^*$ such that $v^*b^* = 0$. Let $Z^* = \{z^* \in B^* | v^*z^* = 0\}$, then Z^* is a right ideal in B^* and hence in K^*. Clearly $Z^* \neq 0$, $Z^* \neq K^*$, and hence by Lemma 3 has dimension 1 or n. Now, since $b^* \in Z^*$, $b^* \notin C^*, Z^* \neq C^*$. Thus, by Lemma 3, Z^* does not have dimension 1. It follows that $Z^* = B^*$ or $v^*x^* = 0$ for all x^* in B^*. Consider now the left ideal $K^* v^*$ of K^*. Any element a^* in K^* can be written $a^* = \alpha 1^* + a^*_1$, where $a^*_1 \in B^*$. Thus, $v^*a^* = \alpha v^*$ or $(K^*v^*)\, a^* \subseteq K^*v^*$ and hence K^*v^* is a two-sided ideal in K^*. Clearly, $C^* \subseteq K^*v^*$ and $K^*v^* \neq K^*$, for otherwise there is an a^* in K^* such that $a^*v^* = 1^*$ or $b^* = a^*v^*b^* = 0$, a contradiction. Thus, by Lemma 3, K^*v^* is either C^* or B^*. Suppose $K^*v^* = B^*$, then $B^* \cdot B^* = K^*v^*B^* = 0$. But this implies that K contains divisors of zero, a contradiction. Thus, $C^* = K^*v^*$ is a two-sided ideal in K^*.

THEOREM 1. *If K is a 𝔊 division algebra of (finite) dimension $n > 2$ over a field F and K^* the associated associative algebra then either K^* is a division algebra or $K^* = F u^* \oplus A^*$ where A^* is a division algebra. In the latter case K is pseudo-equivalent to A^*.*

Proof. Suppose first that K^* contains no right ideals of dimension n. Let C^* be a right ideal of dimension 1 with basis v^*. Clearly there exists a $y^* \notin C^*$ such that $v^*y^* = 0$. Let $Z^* = \{z^* \in K^* | v^*z^* = 0\}$, then Z^* is a right ideal in K^*. Further, $Z^* \neq C^*$, $Z^* \neq 0$, $Z^* \neq K^*$, and hence by Lemma 3 has dimension n contrary to the assumption that K^* contains no right ideals of dimension n. Thus, in this case, K^* is a division algebra.

Suppose next that K^* is not a division algebra and hence contains exactly one right ideal A^* of dimension n. It will be shown that A^* is a division algebra. Thus, suppose that A^* contains a proper right ideal C^*. By Lemma 3 the dimension of C^* is 1. Let $(\alpha_1, f_1), (\alpha_2, f_2), \ldots, (\alpha_n, f_n)$ be a basis for A^* where (α_1, f_1) is a basis for C^*. As in the proof of Lemma 4, $(\alpha_1, f_1)\ (\xi, x) = (0, 0)$ for all (ξ, x) in A^*. Further, since $(\alpha_1, f_1)^2 = (0, 0)$, it follows that $\alpha_1 \neq 0$. Hence A^* has a basis $(1, f), (0, g_2), \ldots, (0, g_n)$ where $(1, f)$ is a basis for C^* and $f, g_2, \ldots, g_n$ span K. Now, by Lemma 4, C^* is a two-sided ideal in A^*. Thus, since A^* contains no proper right ideals other than C^*, it follows that A^*/C^*

contains no proper right ideals. Hence by Lemma 1 A^*/C^* is a division algebra, and has a basis

$$[(0, g_2) + C^*], \ldots, [(0, g_n) + C^*].$$

Clearly A^*/C^* has an identity, say, $[(0, e) + C^*]$, and hence

$$[(0, e) + C^*]\cdot[(0, x) + C^*] = [(0, x) + C^*]$$

for all $[(0, x) + C^*]$ in A^*/C^*. Thus, $(h(e, x), e\cdot x) = (0, x) + \eta(1, f)$, or $ex = x + \eta f$ for all x in the subspace of K spanned by $g_2, \ldots, g_n$. Since $n > 2$, $eg_i = g_i + \eta_i f$, $(i = 2, 3)$, or $ey = y$, where

$$y = \eta_2^{-1}g_2 - \eta_3^{-1}g_3 \neq 0.$$

But, $(1, f)\ (0, y) = (0, 0)$ or $y + fy = 0$, whence $(-f)y = y$. Thus, $e + f = 0$ contrary to the linear independence of $f, g_2, \ldots, g_n$ over F. This proves that A^* contains no proper right ideals and hence, by Lemma 1, is a division algebra. If e^* is the identity in A^* and 1^* the identity in K^*, then $u^* = 1^* - e^*$ is a non-zero idempotent orthogonal to every element of A^* and $u^*, e^*, e^*_2, \ldots, e^*_n$ is a basis for K^* where $e^*, e^*_2, \ldots, e^*_n$ is a basis for A^*. Thus, $K^* = Fu^* \oplus A^*$. Since $1^*, e^*, e^*_2, \ldots, e^*_n$ is also a basis for K^*, it follows that K is pseudo equivalent to A^*.

In connection with Theorem 1 there are two open questions: (i) Is the theorem true when the dimension of K over F is not finite? (ii) Are there any $\mathfrak{G}$ division algebras K for which K^* is a division algebra? This second question indicates that all of the examples of $\mathfrak{G}$ division algebras described in this paper are pseudo-equivalent images of associative division algebras. With this in mind the next theorem is proved.

THEOREM 2. *Let A be an associative division algebra of dimension >2 over a field F and $f(x)$ a non-trivial linear mapping of A into F, with $f(1) = \alpha \neq -1, -\frac{1}{2}$. Let $A(\mathrm{o})$ be the pseudo-equivalent image of A with multiplication defined by $x\mathrm{o}y = f(x)y + f(y)x + xy$. The isotope $A(*)$ of $A(\mathrm{o})$ defined by $x * y = xU^{-1}\mathrm{o}yU^{-1}$, where U is the non-singular linear transformation $x \to x\mathrm{o}1$, is central over F.*

Proof. It follows from the restriction $f(1) = \alpha \neq -1, -\frac{1}{2}$ (in case the characteristic of F is 2, $f(1) \neq 1$ is the requirement) that the linear transformation U is non-singular. Let $\gamma = (2\alpha + 1)^{-1}$, and then a simple computation shows that the product $x * y$ in terms of the multiplication xy in A is given by

$$x * y = (\alpha + 1)^{-2}[xy + \alpha\gamma(f(x)y + f(y)x) - \gamma f(x)f(y)]. \tag{2}$$

Note that $1\mathrm{o}1 = (2\alpha + 1)$ is the identity for $A(*)$.

Suppose first that $\alpha = 0$. Then (2) becomes

$$x * y = xy - f(x)f(y),$$

and the proof that $A(*)$ is central can be found in Albert (**1**, p. 298). Albert's proof is given for an algebra A of finite dimension > 2 over a field F of characteristic 2, but it is easily seen to be valid in this more general case.

To complete the proof, suppose $\alpha \neq 0$, and let c be any element in the centre of $A(*)$. Since the dimension of $A(*) > 2$, there exists an $x \neq 0$ with $f(x) = f(cx) = 0$. Clearly there exists a basis 1, $\{e_\alpha\}$ for A with $f(e_\alpha) = 0$ for all α. Then, since A is a division algebra, x, $\{x\ e_\alpha\}$ is also a basis for A. Suppose that $f(xy) = 0$ for all y in A. Then $f(xe_\alpha) = 0$ for all α, and hence, since $f(x) = 0$, it follows that $f(z) = 0$ for all z in A, a contradiction. Thus, there exists a y such that $f(xy) \neq 0$. With this choice for c, x, y equate $(c * x) * y$ and $c * (x * y)$ to obtain $\alpha\gamma f(xy)c = \gamma f(c)f(xy)$. This implies that $c = \alpha^{-1}f(c)$ is a scalar multiple of the identity in $A(*)$.

2. Some $\mathfrak{S}$ division algebras. Each of the five division algebras described below is obtained in the following way: start with an associative division algebra A over a field F and let $A(\text{o})$ be the pseudo-equivalent algebra with multiplication defined by $x\text{o}y = f(x)y + f(y)x + xy$ for a suitable linear functional $f(x)$ defined on A. Denote by U the linear transformation of A defined by $xU = x\text{o}1$ (1 the identity in A), and let $A(*)$ be the isotope of $A(\text{o})$ defined by $x * y = x\ U^{-1}\text{o}y\ U^{-1}$. In the following examples A and $f(x)$ will be chosen so that $A(\text{o})$ is a division algebra (without identity) and hence $A(*)$ a division algebra with identity $1\text{o}1 = 1 + 2f(1)$. It follows from Theorem 2 that each $A(*)$ is central. The algebras numbered (ii) and (v) appear to be new. References are given for the other three. The following lemma will be used in the construction of each of the five algebras.

LEMMA 5. *Let A be an associative division algebra and $A(\text{o})$ the pseudo-equivalent image with multiplication $x\text{o}y = f(x)y + f(y)x + xy$, where $f(1) \neq -1$. If $A(\text{o})$ contains proper divisors of zero then there exist x, y in $A(\text{o})$ with $f(x) = f(y) = 1$ such that $x\text{o}y = 0$.*

Proof. Choose $x' \neq 0 \neq y'$ so that $x'\text{o}y' = 0$. Clearly $f(x')$, $f(y')$ cannot both be zero. Suppose $f(x') = 0$, then $f(y')x' + x'y' = 0$, or $x'(f(y') + y') = 0$. This implies $y' = -f(y'), f(y') = -f(y')\,f(1)$, or $f(1) = -1$, a contradiction. Thus $f(x') \neq 0$ and similarly $f(y') \neq 0$. Let $x = [f(x')]^{-1}x'$, $y = [f(y')]^{-1}y'$ so that $f(x) = f(y) = 1$. Clearly $x\text{o}y = 0$.

(i) Let F be an ordered field and A a quaternion division algebra over F in which the norm $N(x) = \bar{x}x = x\bar{x}$ is a positive definite quadratic form. Let $f(x) = x + \bar{x}$ and $A(\text{o})$ the pseudo-equivalent image described in Lemma 5. Since $f(1) = 2 \neq -1$, it follows from Lemma 5 that either $A(\text{o})$ is a division algebra or $x\text{o}y = 0$ for some x, y with $f(x) = f(y) = 1$. The latter assumption implies $(1 + x)(1 + y) = 1$. Further, $f(1 + x) = f(1 + y) = 3$, so that $N(1 + x) > 9/4$, $N(1 + y) > 9/4$. Hence

$$1 = N[(1 + x)(1 + y)] = N(1 + x)\cdot N(1 + y) > 81/16,$$

a contradiction. Thus $A(\mathrm{o})$ is a division algebra. The isotope $A(*)$, described at the beginning of this section, closely resembles one of the quasigroup division algebras obtained by Bruck (**2**, p. 179) using the four group.

(ii) Let F be a field of characteristic $\neq 2, 3$, and $F(x)$ the field obtained by adjoining a single indeterminate. Any element $r(x)/q(x)$, where r and q are relatively prime polynomials, may be written in the form

$$\frac{r}{q} = t + \frac{p}{q},$$

where $\deg p < \deg q$ or $p = 0$.

Define

$$f\left(\frac{r}{q}\right) = t(0),$$

the constant term of the polynomial $t(x)$ and note that

$$\frac{r}{q} \to f\left(\frac{r}{q}\right)$$

is a linear mapping of $F(x)$ onto F, with $f(1) = 1$. With this linear functional and $A = F(x)$, define $A(\mathrm{o})$ as in Lemma 5. Since $f(1) = 1 \neq -1$, it follows from Lemma 5 that if $A(\mathrm{o})$ has proper divisors of zero then $a\mathrm{o}b = 0$ for some $a, b \in A$ with $f(a) = f(b) = 1$. Then $(1 + a)(1 + b) = 1$ and $f(1 + a) = f(1 + b) = 2$. Hence

$$1 + a = x\,g(x) + 2 + \frac{p}{q} = t + \frac{p}{q}.$$

But

$$(1 + b) = (1 + a)^{-1} = \frac{q}{tq + p},$$

and if $g(x) \neq 0$, then $\deg q < \deg(tq + p)$ which implies $f(1 + b) = 0$, a contradiction. If $g(x) = 0$, then

$$1 + b = \frac{q}{2q + p}$$

and $f(1 + b) = \frac{1}{2}$. But $2 = \frac{1}{2}$ implies $3 = 0$, a contradiction. Thus, $A(\mathrm{o})$ contains no proper divisors of zero.

To see that the equation $a\mathrm{o}y = b$, $0 \neq a$, $b \in A(\mathrm{o})$, has a solution, first suppose $f(a) = 0$. A simple computation shows that $y = b/a - \frac{1}{2}[f(b/a)]$ is a solution. If $f(a) \neq 0$ it may be assumed that $f(a) = 1$. Then $a \neq -1$ so that $a(1 + a)^{-1}$ exists. Let

$$a = xg(x) + 1 + \frac{p}{q} = t + \frac{p}{q},$$

then

$$1 + a = (1 + t) + \frac{p}{q} = \frac{(1 + t)q + p}{q}$$

and

$$\frac{a}{1+a} = \frac{tq+p}{(1+t)q+p}.$$

Now

$$\gamma = f\left(\frac{a}{1+a}\right) = \begin{cases} 1, & g(x) \neq 0, \\ \frac{1}{2}, & g(x) = 0. \end{cases}$$

In either case $\gamma \neq -1$, so that $(1+\gamma)^{-1}$ exists in F. A simple computation shows that

$$y = \frac{b - (1+\gamma)^{-1} f\left(\dfrac{b}{1+a}\right) a}{1+a}$$

is a solution of the equation $aoy = b$. Since $A(\mathrm{o})$ is commutative and contains no divisors of zero it follows that $A(\mathrm{o})$ is a division algebra. The isotope $A(*)$, as defined above, has an identity $1\mathrm{o}1 = 3$.

(iii) Let F be a field of characteristic two such that there exists a purely inseparable extension field A of dimension $2^r > 2$ and degree 2 over F. Let $x \to f(x)$ be any non trivial linear mapping of A into F such that $f(1) = 0$, and define $A(\mathrm{o})$ as in Lemma 5. Suppose $A(\mathrm{o})$ is not a division algebra. Then, by Lemma 5, $A(\mathrm{o})$ contains x, y with $f(x) = f(y) = 1$ and $xoy = 0$, that is,

$$x + y + x \cdot y = 0. \tag{3}$$

From (3) it follows that $f(xy) = f(x) + f(y) = 1 + 1 = 0$. Multiply (3) on the left by x, and note that $x^2 = \alpha \in F$, to obtain

$$\alpha + xy + \alpha y = 0. \tag{4}$$

From (4) it follows that

$$\alpha = f(\alpha y) = f(\alpha + xy + \alpha y) = f(0) = 0,$$

a contradiction. Thus $A(\mathrm{o})$ is a division algebra. In this example $xU = x\mathrm{o}1 = f(x) + x$, $xU^2 = x$, whence $xU^{-1} = f(x) + x$ and the multiplication $x * y$ in $A(*)$ in terms of the multiplication in A is given by

$$x * y = xU^{-1}\mathrm{o}yU^{-1} = (f(x) + x)\,\mathrm{o}\,(f(y) + y) = f(x)f(y) + xy.$$

The algebra $A(*)$ was constructed by Albert **(1)** who showed that it is a central division algebra, thereby establishing the existence of central commutative division algebras of degree two and characteristic two.

For the next two algebras the following lemma will be needed.

LEMMA 6. *Let A be an associative division algebra of degree 3 over a field F of characteristic $\neq 2$. For $x \in A$ let $\lambda^3 - \tau(x)\lambda^2 + \alpha(x)\lambda - \nu(x) = 0$ be the equation satisfied by x. If $\tau(x) = 1$ then $\tau(x^{-1}) \neq 1$.*

Proof. If $\tau(x) = 1$, then $x^3 - x^2 + \alpha(x)x - \nu(x) = 0$ where $\nu(x) \neq 0$. Multiply by $-[\nu(x)]^{-1}x^{-3}$ to obtain

$$-[\nu(x)]^{-1} + [\nu(x)]^{-1}x^{-1} - \alpha(x)[\nu(x)]^{-1}x^{-2} + x^{-3} = 0.$$

Thus x^{-1} satisfies the equation

$$\lambda^3 - \alpha(x)[\nu(x)]^{-1}\lambda^2 + [\nu(x)]^{-1}\lambda - [\nu(x)]^{-1} = 0,$$

so that $\tau(x^{-1}) = \alpha(x)[\nu(x)]^{-1}$. Suppose $\tau(x^{-1}) = 1$, then $\alpha(x) = \nu(x)$ and the equation satisfied by x is $\lambda^3 - \lambda^2 + \nu(x)\lambda - \nu(x) = 0$. This implies that $x = 1$, whence $\tau(x) = 3$ or $1 = 3$, a contradiction, since the characteristic of F is not 2. Thus $\tau(x^{-1}) \neq 1$.

(iv) Let F be a field of characteristic $\neq 2$, which has a cubic extension A. With $\tau(x)$ defined as in Lemma 6, let $f(x) = -\frac{1}{2}[\tau(x)]$, and note that $f(1) = -3/2$. With this $f(x)$ define $A(\text{o})$ as in Lemma 5. If $A(\text{o})$ is not a division algebra then, by Lemma 5, $A(\text{o})$ contains x and y with $f(x) = f(y) = 1$ and $x\text{o}y = 0$. This implies that $(1 + x)(1 + y) = 1$. But $f(1 + x) = f(1 + y) = -\frac{1}{2}$, so that $\tau(1 + x) = \tau(1 + y) = 1$, contrary to Lemma 6 since $(1 + y) = (1 + x)^{-1}$. The isotope $A(*)$ has an identity $1\text{o}1 = -2$ and has been studied extensively by Dickson **(3)**.

(v) Let A be a cyclic division algebra of degree 3 over a field F of characteristic $\neq 2$. As in the preceeding example let $f(x) = -\frac{1}{2}[\tau(x)]$, $\tau(x)$ defined as in Lemma 6. Again define $A(\text{o})$ as in Lemma 5. The proof that $A(\text{o})$ is a division algebra is the same as the proof above in (iv). The isotope $A(*)$ contains a subalgebra isomorphic to Dickson's algebra of dimension 3 described in (iv).

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A REMARK ON THE ORTHOGONALITY RELATIONS IN THE REPRESENTATION THEORY OF FINITE GROUPS

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Let G be a finite group of order g, and

$$t \to (a_{ij}^{(\mu)}(t)) \qquad (\mu = 1, 2, \ldots, k)$$

be an absolutely irreducible representation of degree f_μ over a field of characteristic zero. As is well known, by using Schur's lemma **(1)**, we can prove the following orthogonality relations for the coefficients $a_{ij}{}^{(\mu)}(t)$:

$$\sum_{t\in G} a_{ij}^{(\mu)}(t)\, a_{kl}^{(\nu)}(t^{-1}) = \delta_{\mu\nu}\delta_{il}\delta_{jk}\frac{g}{f_\mu}. \tag{1}$$

It is easy to conclude from (1) the following orthogonality relations for characters:

$$\sum_{t\in G} \chi^{(\mu)}(t)\chi^{(\nu)}(t^{-1}) = \delta_{\mu\nu}g \tag{2}$$

$$\sum_{\mu=1}^{k} \chi^{(\mu)}(t)\chi^{(\mu)}(s^{-1}) = \bar{\delta}_{t,s}n(t) \tag{3}$$

where

$$\chi^{(\mu)}(t) = \sum_i a_{ii}^{(\mu)}(t),$$

and $\bar{\delta}_{t,s}$ is 1 or 0 according as t and s are conjugate in G or not, and $n(t)$ is the order of the normalizer of t.

In this short note, we remark that we can conclude (1) from (3) or from a special case of (3):

$$\sum_{\mu=1}^{k} f_\mu\chi^{(\mu)}(t) = \delta_{1,t}g. \tag{3'}$$

Let us now asume (3′). Setting $t = 1$ in (3′) we have

$$g = \sum_\mu f_\mu^2.$$

Therefore the number of $(\mu; i, j)$ such that $1 \leqslant \mu \leqslant k$ and $1 \leqslant i, j \leqslant f_\mu$ is g. Let A be the matrix of degree g with the row index t, column index $(\mu; i, j)$ and $(t, (\mu; i, j))$-element $a_{i,j}{}^{(\mu)}(t)$.

Let B be the matrix with row index $(\mu; i, j)$, column index t and $((\mu; i, j), t)$-element

$$\frac{f_\mu}{g}\cdot a_{ji}^{(\mu)}(t^{-1}).$$

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The (t, s)-element in AB is

$$\sum_{\mu, i, j} a^{(\mu)}_{i,j}(t) \cdot \frac{f_\mu}{g} a^{(\mu)}_{ji}(s^{-1})$$

$$= \sum_{\mu, i} \frac{f_\mu}{g} a^{(\mu)}_{ii}(ts^{-1}) = \frac{1}{g} \cdot \sum_{\mu} f_\mu \cdot \chi^{(\mu)}(ts^{-1}) = \delta_{t,s}.$$

This shows that $AB = E$, and hence $BA = E$.

Since the $((\mu; i, j), (\nu; k, l))$-element of BA is

$$\frac{f_\mu}{g} \sum_{t \in G} a^{(\mu)}_{ij}(t)\, a^{(\nu)}_{kl}(t^{-1}),$$

we have (1).

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LINEAR TRANSFORMATIONS ON ALGEBRAS OF MATRICES

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1. Introduction. Let M_n denote the algebra of n-square matrices over the complex numbers; and let U_n, H_n, and R_k denote respectively the unimodular group, the set of Hermitian matrices, and the set of matrices of rank k, in M_n. Let $\mathrm{ev}(A)$ be the set of n eigenvalues of A counting multiplicities. We consider the problem of determining the structure of any linear transformation (l.t.) T of M_n into M_n having one or more of the following properties:

(a) $T(R_k) \subseteq R_k$ for $k = 1, \ldots, n$.
(b) $T(U_n) \subseteq U_n$
(c) $\det T(A) = \det A$ for all $A \in H_n$.
(d) $\mathrm{ev}(T(A)) = \mathrm{ev}(A)$ for all $A \in H_n$.

We remark that we are not in general assuming that T is a multiplicative homomorphism; more precisely, T is a mapping of M_n into itself, satisfying

$$T(aA + bB) = aT(A) + bT(B)$$

for all A, B in M_n and all complex numbers a, b.

We shall show first that if T satisfies property (a), then there exist nonsingular matrices U and V such that either

$$T(A) = UAV$$

or

$$T(A) = UA'V,$$

for all $A \in M_n$, where A' is the transpose of A. We shall then show that any T satisfying (b), (c), or (d) must in turn satisfy (a), and determine the additional restrictions on U and V required in these cases.

2. Rank Preservers. In this section we shall characterize all linear transformations of M_n which preserve rank. To this end it is convenient to consider each matrix of M_n as an n^2-vector, and to represent the l.t. T as an $n^2 \times n^2$ matrix.

$$T = \begin{pmatrix} T_{11} & T_{12} & \ldots & T_{1n} \\ \vdots & & & \\ T_{n1} & & \ldots & T_{nn} \end{pmatrix} \tag{1}$$

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where each T_{ij} is an n-square matrix. If $v_j(A)$ denotes the jth column of A, then T maps $A = (v_1(A), v_2(A), \ldots, v_n(A))$ into the matrix

$$\left(\sum_{j=1}^{n} T_{1j}v_j(A), \ldots, \sum_{j=1}^{n} T_{nj}v_j(A)\right).$$

Let $\rho(A)$ denote the rank of A. If T preserves rank, $T(x, 0, \ldots, 0) = (T_{11}x \ldots, T_{n1}x)$ has rank 1 for any non-zero vector x where 0 is the zero vector. We shall call m n-square matrices $A_1, \ldots, A_m$ *collinear* if, for every non-zero n-vector x,

$$\rho(A_1x, \ldots, A_mx) = 1.$$

LEMMA 1. *If $A_1, \ldots, A_m$ are collinear, there exist non-zero vectors $z_1, \ldots, z_n$* such that

(2) $$v_j(A_i) = k_{ij}z_j, \qquad i = 1, \ldots, m;\ j = 1, \ldots, n;$$

where the k_{ij} are scalars. Moreover, for each j, $k_{ij} \neq 0$ for some i.

Proof. Let e_j denote the unit vector with jth entry equal to 1. Then $A_ie_j = v_j(A_i)$. The lemma follows from the fact that $\rho(v_j(A_1), \ldots, v_j(A_m)) = 1$.

LEMMA 2. *If the matrices $A_1, \ldots, A_m$ are collinear, and z_1, z_β are linearly independent for some β* (cf. (2)), *then there exists a non-singular matrix A and scalars l_i, not all zero, such that*

(3) $$A_i = l_iA, \qquad i = 1, \ldots, m$$

Proof. The matrix $(A_1(e_1 + e_\beta), \ldots, A_m(e_1 + e_\beta)) = (k_{11}z_1 + k_{1\beta}z_\beta, \ldots, k_{m1}z_1 + k_{m\beta}z_\beta)$ has rank 1. For some s, $k_{s1} \neq 0$, by Lemma 1. The Grassmann products

$$(k_{s1}z_1 + k_{s\beta}z_\beta) \wedge (k_{i1}z_1 + k_{i\beta}z_\beta) = 0,$$

for $i = 1, \ldots, m$. Since $z_1 \wedge z_\beta \neq 0$, it follows that $k_{s1}k_{i\beta} - k_{s\beta}k_{i1} = 0$, or

(4) $$k_{i\beta} = \frac{k_{s\beta}k_{i1}}{k_{s1}}, \qquad i = 1, \ldots, m.$$

Moreover, $k_{s\beta} \neq 0$ (otherwise all $k_{i\beta} = 0$); and (4) holds for all β such that z_1 and z_β are independent.

Suppose now that z_1 and z_γ are dependent; then z_β and z_γ are independent. By the preceding argument,

$$k_{i\gamma} = \frac{k_{s\gamma}k_{i\beta}}{k_{s\beta}} = \frac{k_{s\gamma}}{k_{s\beta}}\left(\frac{k_{s\beta}k_{i1}}{k_{s1}}\right) = \frac{k_{s\gamma}k_{i1}}{k_{s1}}, \qquad i = 1, \ldots, m.$$

Thus equations (4) hold for all $1 < \beta < n$. It follows that $A_i = l_iA_s$, $i = 1, \ldots, m$, where $l_i = k_{i1}/k_{s1}$. In particular $l_s = 1$.

The matrix A_s cannot be singular, for then $\rho(A_1x, \ldots, A_mx) = 0$ when x is an eigenvector of A_s corresponding to the eigenvalue 0.

An immediate consequence of Lemmas 1 and 2 is

LEMMA 3. *If the matrices* $A_1, \ldots, A_n$ *are all singular and collinear, then there exist scalars* k_{ij} *and a non-zero vector* z *such that* $v_j(A_i) = k_{ij}z$, $i, j = 1, \ldots, n$.

LEMMA 4. *Let* T *be a rank preserver on* M_n. *If some block* $T_{\alpha\beta}$ *in the representation* (1) *of* T *is non-singular, then there exist scalars* c_{ij} *such that*

$$T_{ij} = c_{ij}T_{\alpha\beta}; \qquad i, j = 1, \ldots, n. \tag{5}$$

Proof. First note that $T_{1\beta}, \ldots, T_{n\beta}$ are collinear. Since $T_{\alpha\beta}$ is non-singular, the vectors $z_1, \ldots, z_n$ of Lemma 1 are linearly independent. Hence $T_{i\beta} = c_{i\beta} T_{\alpha\beta}$, $i = 1, \ldots, n$.

Suppose $T_{\sigma\gamma}$ is also non-singular, $\gamma \neq \beta$. Then $T_{t\gamma} = l_{t\gamma}T_{\sigma\gamma}$, $i = 1, \ldots, n$. If $T_{\sigma\gamma}$ is not a multiple of $T_{\alpha\beta}$, choose a vector x so that $T_{\alpha\beta}x$ and $T_{\sigma\gamma}x$ are linearly independent; and let X be the matrix with $v_j(X) = x$ for $j = \beta, \gamma$, and $v_j(X) = 0$ for $j \neq \beta, \gamma$. Then $\rho(T(X)) = 1$. This implies that

$$(T_{i\beta}x + T_{i\gamma}x) \wedge (T_{t\beta}x + T_{t\gamma}x) = 0, \qquad i, t = 1, \ldots, n.$$

Since $T_{\alpha\beta}x \wedge T_{\sigma\gamma}x \neq 0$,

$$c_{i\beta}l_{t\gamma} - l_{i\gamma}c_{t\beta} = 0 \text{ for all } i, t. \tag{6}$$

Let Y be a matrix for which $v_\beta(Y)$ and $v_\gamma(Y)$ are independent and $v_j(Y) = 0$ for $j \neq \beta, \gamma$. Then $\rho(Y) = 2$, while $\rho(T(Y)) \leq 1$ by (6). This contradiction shows that $T_{\sigma\gamma}$ is a multiple of $T_{\alpha\beta}$, and (5) holds for $T_{i\gamma}$, $i = 1, \ldots, n$.

Finally suppose that $T_{i\gamma}$ is singular for some γ and all i. By Lemma 3 there exist scalars k_{ij} and a non-zero vector z such that $v_j(T_{i\gamma}) = k_{ij}z$. Thus $T_{i\gamma}x$ is a multiple of z for any vector x. Choose y so that $T_{\alpha\beta}y = z$, and choose x independent of y. For the matrix Y above with $v_\beta(Y) = y$ and $v_\gamma(Y) = x$, $\rho(Y) = 2$, while $\rho(T(Y)) \leq 1$. Hence this case cannot arise. This completes the proof of the lemma.

Not every rank preserver need have a non-singular block in its representation (1). For example, the transformation T_1, which maps each matrix onto its transpose, is linear and preserves rank. In its matrix, $T_{ij} = E_{ji}$, where E_{ij} is the matrix with 1 in the i, j position and 0's elsewhere. We have, however, the following result.

LEMMA 5. *Let* T *be a rank preserver. If every* T_{ij} *in the representation* (1) *is singular, then the* $n^2 \times n^2$ *matrix* TT_1 *has a non-singular block.*

Proof. By Lemma 3, there exist vectors $z_1, \ldots, z_n$ such that each column of T_{ij} is a multiple of z_j for $i, j = 1, \ldots, n$. For any matrix A, $v_i(T(A))$ is a linear combination of the columns of the T_{ij}. Hence the columns of $T(A)$ are linear combinations of the vectors z_j. This implies that $z_1, \ldots, z_n$ are linearly independent; for, if not, the columns of $T(A)$ would be linearly dependent, which is not the case when A is non-singular. Denote the blocks of TT_1 by W_{ij}, $i, j = 1, \ldots, n$. Then

$$W_{ij} = \sum_{k=1}^{n} T_{ik}E_{jk},$$

and $v_k(W_{ij}) = v_j(T_{ik})$. Thus the kth column of each W_{ij} is a multiple of z_k. Since TT_1 preserves rank, the blocks $W_{11}, \ldots, W_{n1}$ are collinear. The result then follows from Lemma 2.

THEOREM 1. *Let T be a* l.t. *of M_n into M_n. T is a rank preserver if and only if there exist non-singular matrices U and V such that either*:

(7) $$T(A) = UAV \quad \text{for all } A,$$

or

(8) $$T(A) = UA'V \quad \text{for all } A.$$

Proof. The sufficiency of the condition is obvious. For the necessity, if the representation (1) of T has a non-singular block $T_{\alpha\beta}$, choose $U = T_{\alpha\beta}$ and $V = (c_{ji})$ in Lemma 4. If T has no non-singular block, define the rank preserver T_2 by $T_2(A') = T(A)$. By Lemma 5, T_2 has a non-singular block; hence there exist U and V non-singular such that $T(A) = T_2(A') = UA'V$ for all A.

3. Determinant Preservers. We shall show that, if a linear transformation T of M_n maps unimodular matrices into unimodular matrices, it preserves determinant; that if it preserves determinant, it preserves rank; and determine the appropriate forms of U and V in Theorem 1.

LEMMA 6. *If the* l.t. *T maps U_n into U_n, then* $\det T(A) = \det A$ *for all matrices A.*

Proof. If $\det A \neq 0$, $\det [A/(\det A)^{1/n}] = 1$; hence $\det T(A) = (\det A)$. $\det [T(A/(\det A)^{1/n})] = \det A$. Now $\det T(A)$ is a polynomial in the entries a_{ij} of A which is equal to $\det A$ for all non-singular A; thus this relation is an identity so that $\det T(A) = \det A$ for all A.

LEMMA 7. *If T preserves determinant, then T is non-singular and hence onto.*

Proof. Suppose $T(A) = 0$; then $\rho(A) < n$. There exist non-singular matrices M and N such that $MAN = I_r \dotplus 0_{n-r}$, where $r = \rho(A)$, I_r is the $r \times r$ unit matrix, 0_{n-r} is the $(n-r) \times (n-r)$ zero matrix and $\dotplus$ denotes the direct sum. For any X, $[\det (MAN + X)]/\det MN = \det (A + M^{-1}XN^{-1}) = \det T(A + M^{-1}XN^{-1}) = \det T(M^{-1}XN^{-1}) = \det M^{-1}XN^{-1}$. Hence $\det (MAN + X) = \det X$. Set $X = 0_r \dotplus I_{n-r}$. Then $\det (MAN + X) = 1$, while $\det X = 0$ unless $r = 0$. Hence $A = 0$.

LEMMA 8. *If T preserves determinant, then T preserves rank.*

Proof. Let A be an arbitrary matrix. There exist non-singular matrices M_1, N_1, M_2, N_2, such that $M_1AN_1 = Y_1 = I_r \dotplus 0_{n-r}$ and $M_2T(A)N_2 = Y_2 = I_s \dotplus 0_{n-s}$ where $r = \rho(A)$ and $s = \rho(T(A))$. Define a mapping ϕ of M_n by:

$$\phi(X) = M_2T(M_1^{-1}XN_1^{-1})N_2.$$

Then ϕ is linear with the property: $\det \phi(X) = k \det X$, where

$$k = \det (M_2 M_1^{-1} N_1^{-1} N_2);$$

also $\phi(Y_1) = Y_2$. Set $Y_3 = 0_r \dotplus I_{n-r}$. For any scalar λ, $\det (\lambda Y_1 + Y_3) = \lambda^r$. On the other hand, $\det \phi(\lambda Y_1 + Y_3) = \det (\lambda Y_2 + \phi(Y_3)) = p(\lambda)$, a polynomial in λ of degrees $\leq s$. Since $p(\lambda) \equiv k\lambda^r$, $k \neq 0$, identically in λ, it follows that $r \leq s$, and $\rho(A) \leq \rho(T(A))$.

By Lemma 7, T^{-1} exists; moreover, since T preserves determinant, $\det B = \det (TT^{-1}(B)) = \det T^{-1}(B)$ for all B in M_n. Thus T^{-1} preserves determinant, and $\rho(T(A)) \leq \rho(T^{-1}T(A)) = \rho(A)$. Therefore $\rho(A) = \rho(T(A))$.

Theorem 2. *Let T be a* l.t. *of M_n. The following conditions are equivalent:*

(i) *T maps U_n into U_n.*

(ii) *T preserves determinant.*

(iii) *There exist unimodular matrices U and V such that either* (7) *or* (8) *holds.*

Proof. Lemma 6 gives (i) $\leftrightarrow$ (ii); (iii) $\rightarrow$ (ii) is obvious. If T preserves determinant, then by Lemma 8 and Theorem 1, there exist non-singular matrices U_1 and V_1 such that $T(A) = U_1 A V_1$ or $T(A) = U_1 A' V_1$. Since $\det T(I) = 1$, $\det U_1 V_1 = 1$. Choose $U = U_1/(\det U_1)^{1/n}$ and $V = V_1/(\det V_1)^{1/n}$. Thus (ii) $\rightarrow$ (iii).

We shall show in the next section that preservation of determinant for Hermitian matrices is also equivalent to conditions (i)—(iii).

4. Eigenvalue Preservers.

Lemma 9. *Let T be a* l.t. *of M_n. If* $\mathrm{ev}(T(H)) = \mathrm{ev}\,(H)$ *for all Hermitian matrices H, then* $\mathrm{ev}(T(A)) = \mathrm{ev}(A)$ *for all A in M_n.*

Proof. Note first that if H is Hermitian and satisfies the given condition, then $\mathrm{tr}\{[T(H)]^m\} = tr\{H^m\}$ for $m = 1, 2, \ldots$, where $tr(X)$ denotes the trace of X. For any matrix A there exist Hermitian matrices K, L such that $A = K + iL$. For real α, $K + \alpha L$ is Hermitian and

$$tr\{[T(K + \alpha L)]^m\} = tr\{(K + \alpha L)^m\}. \tag{9}$$

For each m, equation (9) is a polynomial equation in α of degree $\leq m$ satisfied by all real α. Hence (9) is satisfied by all complex α, and in particular by $\alpha = i$. If the eigenvalues of A and $T(A)$ are λ_j and μ_j, respectively, $j = 1, \ldots, n$, then

$$\sum_{j=1}^{n} \lambda_j^m = \sum_{j=1}^{n} \mu_j^m \qquad m = 1, 2, \ldots.$$

It follows that the corresponding elementary symmetric functions of the λ_j and the μ_j are equal, and that $\mathrm{ev}(T(A)) = \mathrm{ev}(A)$.

LEMMA 10. *If* $\text{ev}(T(A)) = \text{ev}(A)$ *for all* $A \in M_n$, *then* $T(I) = I$, *where* I *is the unit matrix of order* n.

Proof. T preserves determinant. Hence, for λ complex and $A \in M_n$, $\det(\lambda I - A) = \det(\lambda T(I) - T(A)) = \det(\lambda I - CT(A))$, where $C = (T(I))^{-1}$. Thus $\text{ev}(T(A)) = \text{ev}(A) = \text{ev}(CT(A))$. Since T is non-singular by Lemma 7, $T(A)$ ranges over all of M_n as A does. Hence $\text{ev}(A) = \text{ev}(CA)$ for all A. Choose U unitary so that $CU = H$, where H is positive definite Hermitian. Then $\text{ev}(U) = \text{ev}(CU) = \text{ev}(H)$, so that U has positive eigenvalues. Hence $U = 1$ and $C = H$. Since the eigenvalues of $C = (T(I))^{-1}$ are all 1, $C = I$, $C^{-1} = I$, and $T(I) = I$.

From Lemmas 9 and 10 and Theorem 2 we obtain

THEOREM 3. *Let* T *be a* l.t. *of* M_n. *The following conditions are equivalent*:

(i) *T preserves eigenvalues for all Hermitian matrices in* M_n.

(ii) *T preserves eigenvalues for all matrices in* M_n.

(iii) *There exists a unimodular matrix* U *such that either* $T(A) = UAU^{-1}$ *for all* $A \in M$ *or* $T(A) = UA'U^{-1}$ *for all* $A \in M_n$.

THEOREM 4. *Let* T *be a* l.t. *of* M_n. *If* $\text{ev}(T(H)) = \text{ev}(H)$ *and* $T(H)$ *is Hermitian for all Hermitian* H *in* M_n, *then the matrix* U *in Theorem* 3 (iii) *is unitary*.

Proof. $T(H) = (T(H))^*$ implies $UHU^{-1} = U^{-1*}HU^*$ and $U^*UH = HU^*U$. for all Hermitian H. It follows easily that $U^*U = I$.

THEOREM 5. *Let* T *be a* l.t. *of* M_n. *Then* T *preserves determinant if and only if it preserves determinant for Hermitian matrices.*

Proof. Define $\phi(A) = CT(A)$, where $C = (T(I))^{-1}$. If T preserves determinant for Hermitian H, then $\det(\lambda I - H) = \det(\lambda T(I) - T(H)) = \det(\lambda I - \phi(H))$ for all real λ. Hence $\text{ev}(\phi(H)) = \text{ev}(H)$, and by Lemma 9, $\text{ev}(\phi(A)) = \text{ev}(A)$ for all A. Thus $\det A = \det \phi(A) = \det \text{T}(A)$ for all A.

Professor N. Jacobson communicated to us the following information while this paper was in press: Theorem 1 was obtained by L. K. Hua (Science Reports of the National Tsing Hua University, Ser. A, *5* (1948) pp. 150–81) and in more general form by H. Jacob (Amer. J. Math., **77** (1955) pp. 177–89). In both these papers T is assumed non-singular; actually our proof of Theorem 1 requires only that $T(R_i) \subseteq R_i$, for $i = 1, 2, n$ without the assumption that T be non-singular. Also Dieudonné (Archiv. d. Math., *1* (1948) pp. 282–7) shows that if T preserves the cone $\det A = 0$ and T is non-singular then T has the form indicated in Theorem 2 (iii). Again, our result does not require the assumption that T be non-singular: this follows if T preserves all determinants (Lemma 7).

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ON FREE PRODUCTS OF CYCLIC ROTATION GROUPS

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We consider the group of rotations in three-dimensional Euclidean space, leaving the origin fixed. These rotations are represented by real orthogonal third-order matrices with positive determinant. It is known that this rotation group contains free non-abelian subgroups of continuous rank (see **1**).

In this paper we shall prove the following conjectures of J. de Groot (**1**, pp. 261–262):

THEOREM 1. *Two rotations with equal rotation angles α and with arbitrary but different rotation axes are free generators of a free group, if $\cos\alpha$ is transcendental.*

THEOREM 2. *A free product of at most continuously many cyclic groups can be isomorphically represented by a rotation group.*

More precisely: Theorem 2 is a special case of the following conjecture of J. de Groot (**1**, p. 262): A free product of at most continuously many rotation groups, each consisting of less than continuously many elements, can be isomorphically represented by a rotation group.

J. Mycielski at Wroclaw informed me that he, with S. Balcerzyk has proved a theorem, which includes our Theorem 2 as a special case; moreover, our Theorem 1 seems to intersect with a theorem proved by S. Balcerzyk.

Preliminaries. We define

$$A(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}, \qquad D(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $i_\sigma = \pm 1$, $k_\sigma = 1, 2, 3, \ldots$ $(\sigma = 1, \ldots, s)$. Furthermore, we assume $\cos\alpha$ is transcendental. Then we have

LEMMA: *No product P_s $(s > 1)$ of the form*

$$P_s = D(\theta_0)A^{i_1k_1}(\alpha)D(\theta_1)A^{i_2k_2}(\alpha)\ldots A^{i_sk_s}(\alpha)D(\theta_s)$$

is the identity, if one of the following conditions is satisfied for $\sigma = 1, \ldots, s-1$:

(a) *θ_σ is not a multiple of π;*

(b) *θ_σ is not a multiple of 2π and the exponents of A are of alternating sign: $i_{\sigma+1} = -i_\sigma$.*

Proof: We use the formulas $(k > 0)$:

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$$\cos k\alpha = 2^{k-1}\cos^k\alpha + \dots,$$
$$\sin k\alpha = \sin\alpha\,(2^{k-1}\cos^{k-1}\alpha + \dots),$$
$$\sin^2\alpha = 1 - \cos^2\alpha,$$

where ... denote terms of lower degree in $\cos\alpha$, So we have

(1) $A^{i_\sigma k_\sigma}(\alpha)D(\theta_\sigma) =$

$$\begin{pmatrix} \cos\theta_\sigma, & -\sin\theta_\sigma, & 0 \\ q_\sigma \sin\theta_\sigma \cos\alpha + \dots, & q_\sigma \cos\theta_\sigma \cos\alpha + \dots, & -i_\sigma q_\sigma \sin\alpha + \dots \\ i_\sigma q_\sigma \sin\theta_\sigma \sin\alpha + \dots, & i_\sigma q_\sigma \cos\theta_\sigma \sin\alpha + \dots, & q_\sigma \cos\alpha + \dots \end{pmatrix},$$

where ... denote terms of lower degree in $\cos\alpha$ and $\sin\alpha$ and

$$q_\sigma = (2\cos\alpha)^{k_\sigma - 1}.$$

By induction with respect to σ we find that the elements of the matrices $P_\sigma = (p^\sigma{}_{ik})$ are polynomials in $\cos\alpha$, multiplied or not by a factor $\sin\alpha$. In particular the elements $p^\sigma{}_{32}$ and $p^\sigma{}_{33}$ obtain the form (we consider the leading terms only, denoting terms of lower degree by ...):

$$p^\sigma{}_{32} = i_\sigma q_\sigma V_\sigma \cos\theta_\sigma \sin\alpha + \dots,$$
$$p^\sigma{}_{33} = q_\sigma V_\sigma \cos\alpha + \dots.$$

Indeed, for $\sigma = 1$ we have $V_1 = 1$ and multiplying P_σ with the matrix (1), where σ is replaced by $\sigma + 1$, we find

$$\begin{aligned} p_{32}^{\sigma+1} &= i_\sigma q_\sigma V_\sigma \cos\theta_\sigma \sin\alpha \cdot q_{\sigma+1} \cos\theta_{\sigma+1} \cos\alpha \\ &\quad + q_\sigma V_\sigma \cos\alpha \cdot i_{\sigma+1} q_{\sigma+1} \cos\theta_{\sigma+1} \sin\alpha + \dots \\ &= i_{\sigma+1} q_{\sigma+1} q_\sigma V_\sigma \cos\theta_{\sigma+1} \cos\alpha \sin\alpha\,(i_\sigma i_{\sigma+1} \cos\theta_\sigma + 1) + \dots \\ p_{33}^{\sigma+1} &= i_\sigma q_\sigma V_\sigma \cos\theta_\sigma \sin\alpha \cdot -\, i_{\sigma+1} q_{\sigma+1} \sin\alpha \\ &\quad + q_\sigma V_\sigma \cos\alpha \cdot q_{\sigma+1} \cos\alpha + \dots \\ &= q_\sigma q_{\sigma+1} V_\sigma \cos^2\alpha (i_\sigma i_{\sigma+1} \cos\theta_\sigma + 1) + \dots. \end{aligned}$$

Hence,

$$V_{\sigma+1} = q_\sigma V_\sigma \cos\alpha\,(1 + i_\sigma i_{\sigma+1} \cos\theta_\sigma).$$

From this it follows that the coefficient of the leading term of $p^s{}_{33}$ does not vanish if

$$1 + i_\sigma i_{\sigma+1} \cos\theta_\sigma \neq 0 \qquad (\sigma = 1, \dots, s-1),$$

that is, if (a) or (b) holds true.

Thus since $p^s{}_{33}$ is a polynomial in $\cos\alpha$ and $\cos\alpha$ is transcendental, the product P_s satisfying (a) or (b) obviously is unequal to the identity, by which the lemma is proved.

Proof of Theorem 1. Two rotations with rotation angles α, the axes of which intersect under an angle θ, may be represented by the matrices $A = A(\alpha)$ and $B = D(\theta)A(\alpha)D(-\theta)$. Clearly the theorem is proved if we show that A and B generate a free non-abelian group when $\cos\alpha$ is transcendental and θ is not a multiple of π.

Since all non-trivial products of the elements $A^{\pm 1}$ and $B^{\pm 1}$ have the form P_s satisfying condition (a), they are not equal to the identity by virtue of the lemma, by which the theorem is proved.

Proof of Theorem 2. J. von Neumann **(2)** proved that the real numbers x_t defined by

$$x_t = \sum_{n=0}^{\infty} 2^{2^{[nt]}-2^{n^2}} \qquad (t > 0)$$

are algebraically independent over the field of rational numbers.

We define

$$(2) \qquad \begin{cases} \phi_t = 2 \operatorname{arctg} x_t \\ \alpha = 2 \operatorname{arctg} x_1 \end{cases} \qquad (0 < t < 1).$$

Then, according to a theorem of J. de Groot **(1)**, we have:

The continuously many rotations

$$B_t = D(\phi_t)A(\alpha)D(-\phi_t) \qquad (0 < t < 1)$$

are free generators of a free rotation group.

Let (F) denote the group generated by the rotation F. We shall now prove:

The group generated by the continuously many rotations

$$F_t(\delta_t) = B_t D(\delta_t) B_t^{-1} \qquad (0 < t < 1)$$

is a free product of the cyclic groups $(F_t(\delta_t))$. *This obviously implies Theorem* 2.

Consider any non-trivial product

$$\begin{aligned} &F_{t_1}^{m_1}(\delta_{t_1})F_{t_2}^{m_2}(\delta_{t_2}) \ldots F_{t_s}^{m_s}(\delta_{t_s}) \\ &\quad = D(\phi_{t_1})A(\alpha)D(m_1\delta_{t_1})A^{-1}(\alpha)D(\phi_{t_2} - \phi_{t_1})A(\alpha)D(m_2\delta_{t_2}) \ldots \\ &\qquad A(\alpha)D(m_s\delta_{t_s})A^{-1}(\alpha)D(-\phi_{t_s}). \end{aligned}$$

We may assume that

$$m_k\delta_{t_k} \qquad (k = 1, \ldots, s)$$

is not a multiple of 2π, for otherwise we have a trivial product. Furthermore, the numbers

$$\phi_{t_{k+1}} - \phi_{t_k} \qquad (k = 1, \ldots, s-1)$$

are not multiples of 2π by virtue of (2). Thus the product considered has the form P_{2s} satisfying condition (b). According to the lemma this product is unequal to the identity, by which the theorem is proved.

References

1. J. de Groot, *Orthogonal isomorphic representations of free groups*, Can. J. Math., *8* (1956), 256–262.
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CHARACTERS OF CARTESIAN PRODUCTS OF ALGEBRAS

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Introduction. Let R be a commutative ring with identity 1. A *character* of an R-algebra E is a homomorphism from E onto R, regarded as an algebra over itself. If $(E_\alpha)_{\alpha \in A}$ is a family of R-algebras indexed by a set A and if

$$E = \prod_{\alpha \in A} E_\alpha,$$

then for every $\beta \in A$ and every character v_β of E_β, $v_\beta \circ pr_\beta$ is a character of E where pr_β is the projection homomorphism from E onto E_β. Further if A is finite and if the only idempotents of R are 0 and 1 (equivalently, if R is not the direct sum of two proper ideals), it is easy to see that every character of E is of this form. In general, it is natural to ask:

(1) *Is every character of*

$$E = \prod_{\alpha \in A} E_\alpha$$

of the form $v_\beta \circ pr_\beta$ for some $\beta \in A$, where v_β is a character of E_β?

If each E_α is R, E is simply the R-algebra of all R-valued functions with domain A; we shall denote this algebra by R^A, the set of its characters by $M(R^A)$, and its identity element by e. Since the only character of the R-algebra R is the identity map, (1) becomes for R^A:

(2) *Is every character of R^A a projection?*

Question (1) appears more general than (2), but we shall see in § 1, as a consequence of an extension theorem of Buck, that an affirmative answer to (2) implies an affirmative answer to (1).

Recently, by a measure-theoretic argument, Bialynicki-Birula and Żelazko **(1)** answered (1) in the affirmative if R is an infinite field, if each E_α has an identity, and if A satisfies a certain set-theoretic condition. The author obtained his results independently (without the hypothesis that each E_α possess an identity) as corollaries of a density theorem concerning a suitable weak uniform structure imposed on the set of characters of R^A. These results are given in §§ 2 and 3. In §§ 4 and 5 we shall prove that if R is finite and A infinite, question (2) has a negative answer, but that if R is a principal domain having at least two non-associated extremal elements (for example, if R is the integers) and if A satisfies a certain set-theoretic condition, the

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questions have an affirmative answer. These results are applied in the remaining two sections: in § 6 we show that the only compact principal domains are the known ones, namely, finite fields and valuation rings of locally compact fields whose topology is given by a discrete valuation of rank 1; and in § 7 we give conditions on the algebra $\mathfrak{C}(T)$ of all real-valued continuous functions on topological space T which are both necessary and sufficient for every connected component of T to be open.

1. The extension theorem. Buck's extension theorem **(5**, Theorem 1**)** may be stated in its most general form as follows (as observed in **(8**, p. 74**)**, one of the hypotheses of Buck's original version is superfluous).

THEOREM A. *Let R be a commutative ring with identity, E an R-algebra, H an ideal of E, F an R-algebra with identity. If f is a homomorphism from H onto F, there exists a unique homomorphism g from E onto F extending f.*

Consequently, we see that an affirmative answer to question (2) implies an affirmative answer to question (1):

THEOREM 1. *Let $(E_\alpha)_{\alpha\in A}$ be a family of R-algebras indexed by A. If every character of R^A is a projection, then for every character u of*

$$E = \prod_{\alpha\in A} E_\alpha$$

there exist $\beta \in A$ and a character v_β of E_β such that $u = v_\beta \circ pr_\beta$.

Proof. First, let us assume each E_α has an identity e_α. The restriction of u to the subalgebra

$$F = \prod_{\alpha\in A} Re_\alpha$$

of E is a character of F since F contains the identity of E. As F is canonically isomorphic with R^A, it follows from the hypothesis that there exists $\beta \in A$ such that $u(i_\beta(e_\beta)) = 1$, where i_β is the canonical injection map from E_β into E. Hence if $v_\beta = u \circ i_\beta$, v_β is a character of E_β, and u and $v_\beta \circ pr_\beta$ coincide on the ideal $i_\beta(E_\beta)$ of E. Therefore, as $v_\beta \circ pr_\beta$ is a character of E, the uniqueness part of Buck's theorem ensures $u = v_\beta \circ pr_\beta$. In the general case, let E_α^+ be the R-algebra obtained by adjoining an identity to E_α. As E_α is an ideal in E_α^+, E is an ideal in

$$G = \prod_{\alpha\in A} E_\alpha^+.$$

By Buck's theorem, there exists a character of G extending u, and an application of the preceding result completes the proof.

2. Algebras over fields. Let K be a field equipped with the discrete topology. K is then a topological field whose associated uniform structure is the discrete uniform structure. Let $\mathfrak{U}_K(A)$ be the weakest uniform structure on A such

that each $f \in K^A$ is uniformly continuous, let $\mathfrak{W}_K(A)$ be the weakest uniform structure on the set $\mathfrak{F}(K^A, K)$ of all K-valued functions on K^A such that $u \to u(f)$ is uniformly continuous on $\mathfrak{F}(K^A, K)$ for all $f \in K^A$, and let $\mathfrak{V}_K(A)$ be the uniform structure induced on $M(K^A)$ by $\mathfrak{W}_K(A)$. As the uniform structure of K is complete and separated, $\mathfrak{W}_K(A)$ is a complete, separated uniform structure (**4**, § 1, Proposition 2, and Theorem 1). A familiar argument shows $M(K^A)$ is closed: if $\mathfrak{F}$ is a filter on $M(K^A)$ converging to $u \in \mathfrak{F}(K^A, K)$ and if $f, g \in K^A$, then $\mathfrak{F}(fg) \to u(fg)$, $\mathfrak{F}(f) \to u(f)$, and $\mathfrak{F}(g) \to u(g)$; for any $F \in \mathfrak{F}$, $(fg)(F) \subseteq f(F)g(F)$, so $\mathfrak{F}(fg)$ is the filter base for a filter finer than that generated by $\mathfrak{F}(f)\cdot\mathfrak{F}(g)$; hence as $\mathfrak{F}(f)\cdot\mathfrak{F}(g) \to u(f)u(g)$, so also $\mathfrak{F}(fg) \to u(f)u(g)$, and therefore $u(fg) = u(f)u(g)$. Similarly, u is linear. $\mathfrak{F}(e) \to u(e)$, but as $v(e) = 1$ for all $v \in M(K^A)$, $u(e) = 1$. Hence $u \in M(K^A)$. $M(K^A)$ is therefore a complete separated uniform space. For any finite subset Γ of K^A, let $U(\Gamma) = [(\alpha, \beta) \in A \times A : f(\alpha) = f(\beta)$ for all $f \in \Gamma]$, $V(\Gamma) = [(u, v) \in M(K^A) \times M(K^A): u(f) = v(f)$ for all $f \in \Gamma]$. The collection of sets $U(\Gamma)$ [respectively, $V(\Gamma)$] forms a fundamental system of entourages for $\mathfrak{U}_K(A)$ [respectively, $\mathfrak{V}_K(A)$] as Γ ranges through all finite subsets of K^A. For each $\alpha \in A$ let $\alpha^\wedge$ be the projection $f \to f(\alpha)$ on K^A. Then $\alpha \to \alpha^\wedge$ is clearly a uniform structure isomorphism from A into $M(K^A)$, and we shall denote by $A^\wedge$ the image of A under this map.

THEOREM 2. *For any field K, $A^\wedge$ is dense in $M(K^A)$.*

Proof. Let $u \in M(K^A)$; we shall prove there exists a filter on $A^\wedge$ converging to u. Let H be the kernel of u, a proper ideal of K^A. For each finite subset Γ of H let $F(\Gamma) = [\alpha^\wedge \in A^\wedge : f(\alpha) = 0$ for all $f \in \Gamma]$. Clearly $F(\Gamma_1) \cap F(\Gamma_2) = F(\Gamma_1 \cup \Gamma_2)$, so to prove the sets $F(\Gamma)$ form a filter base for a filter $\mathfrak{F}$ on $A^\wedge$, it suffices to prove $F(\Gamma) \neq \phi$ for all finite subsets Γ of H. Suppose $F(\Gamma) = \phi$ for some $\Gamma = \{f_1, \ldots, f_n\} \subseteq$ H; we define $g_1, \ldots, g_n$ inductively by letting $g_1 = e$ and, for $j > 1$, letting g_j be the characteristic function of

$$\left[\alpha \in A : \left(\sum_{k=1}^{j-1} f_k g_k\right)(\alpha) = 0\right].$$

Then if $h = \Sigma_1^n f_j g_j$, $h \in H$ and, since $F(\Gamma) = \phi$, $h(\alpha) \neq 0$ for all $\alpha \in A$. But then $e = h \cdot (e/h) \in H$, so $H = K^A$ which is impossible. Finally, the filter $\mathfrak{F}$ thus defined converges to u: if $\Gamma = \{f_1, \ldots, f_n\} \subseteq K^A$, for each j let $h_j = f_j - u(f_j)e$; then $\Gamma_0 = \{h_1, \ldots, h_n\} \subseteq H$. $F(\Gamma_0) \subseteq V(\Gamma)(u)$, for if $\alpha^\wedge \in F(\Gamma_0)$ and if $1 \leqslant j \leqslant n$, $\alpha^\wedge(f_j) - u(f_j) = \alpha^\wedge(h_j + u(f_j)e) - u(f_j) = \alpha^\wedge(h_j) = h_j(\alpha) = 0$, by definition of $F(\Gamma_0)$. Hence $\mathfrak{F} \to u$, and the proof is complete.

COROLLARY 1. *If K is a field, every character of K^A is a projection if and only if $\mathfrak{U}_K(A)$ is complete.*

COROLLARY 2. *If K is a field, $(E_\alpha)_{\alpha \in A}$ a family of K-algebras indexed by A, and if $\mathfrak{U}_K(A)$ is complete, then for every character v of $E = \Pi_\alpha E_\alpha$ there exist $\beta \in A$ and a character v_β of E_β such that $v = v_\beta \circ pr_\beta$.*

3. The theorems of Bialynicki-Birula and Żelazko. An *Ulam measure* on set A is a non-zero, countably additive set-function λ, defined on the class of all subsets of A, taking on only the values 0 and 1, such that $\lambda(X) = 0$ for all finite subsets X of A; an *Ulam ultrafilter* on A is an ultrafilter $\mathfrak{U}$ such that the intersection of any countable subfamily of $\mathfrak{U}$ is again a member of $\mathfrak{U}$; a *point ultrafilter* on A is simply the (Ulam) ultrafilter of all subsets of A containing a given point of A. If λ is an Ulam measure, the sets X such that $\lambda(X) = 1$ form an Ulam ultrafilter which is not a point ultrafilter; conversely any Ulam ultrafilter which is not a point ultrafilter defines an Ulam measure. Thus A admits no Ulam measure if and only if every Ulam ultrafilter on A is a point ultrafilter.

Bialynicki-Birula and Żelazko **(1)** proved the following results (under the additional hypothesis in Theorem B that each E_α possessed an identity):

THEOREM B. *Let K be an infinite field, $(E_\alpha)_{\alpha \in A}$ a family of K-algebras indexed by a set A which either admits no Ulam measure or has cardinality not greater than that of K. Then for every character u of $E = \Pi_\alpha E_\alpha$, there exist $\beta \in A$ and a character v_β of E_β such that $u = v_\beta \circ pr_\beta$.*

THEOREM C. *If K is an infinite field admitting no Ulam measure, then A admits no Ulam measure if and only if every character of K^A is a projection.*

By Corollary 2 of Theorem 1, to prove Theorem B it suffices to show that either of its hypotheses concerning A ensures $\mathfrak{U}_K(A)$ is complete. If the cardinality of A is not greater than that of K, there exists a one-to-one function $g \in K^A$. $U(\{g\})$ is then the diagonal in $A \times A$, so $\mathfrak{U}_K(A)$ is the discrete uniform structure and hence is complete. Suppose A admits no Ulam measure. Let $\mathfrak{F}$ be a Cauchy filter on A and let $\mathfrak{U}$ be an ultrafilter containing $\mathfrak{F}$. $\mathfrak{U}$ is an Ulam ultrafilter: let $(\lambda_n)_{n>0}$ be a sequence of distinct non-zero elements of K. If $(F_n)_{n>0}$ is any decreasing sequence of members of $\mathfrak{U}$ such that $F_0 = A$, let $g(\alpha) = \lambda_n$ for all $\alpha \in F_n - F_{n+1}$, $g(\alpha) = 0$ for all $\alpha \in F = \bigcap_{n>0} F_n$, and let $C \in \mathfrak{U}$ be $U(\{g\})$-small. If $C \cap (F_n - F_{n+1}) \neq \phi$, $C \cap F_{n+1} = \phi$ by definition of g, which is impossible. Hence $C \subseteq F$, so $F \in \mathfrak{U}$. Thus by hypothesis, as every Ulam ultrafilter is a point ultrafilter, there exists $\beta \in A$ which is contained in each member of $\mathfrak{U}$. β is then an adherent point of the Cauchy filter $\mathfrak{F}$, so $\mathfrak{F}$ converges to β and the proof is complete.

To prove Theorem C, it suffices by Theorem B and Corollary 1 of Theorem 2 to show that if $\mathfrak{U}_K(A)$ is complete and if every Ulam ultrafilter on K is a point ultrafilter, then every Ulam ultrafilter $\mathfrak{U}$ on A is a point ultrafilter. For each $f \in K^A$, $[L \subseteq K : f^{-1}(L) \in \mathfrak{U}]$ is clearly an Ulam ultrafilter on K, so there exists $\lambda \in K$ such that $f^{-1}(\lambda) \in \mathfrak{U}$. But $f^{-1}(\lambda)$ is $U(\{f\})$-small; it follows easily that $\mathfrak{U}$ is a Cauchy filter on A and therefore converges. As the topology defined by $\mathfrak{U}_K(A)$ is the discrete topology, $\mathfrak{U}$ is therefore a point ultrafilter.

4. Algebras over finite rings. We next ask for what other commutative rings R with identity does question (2) (and therefore question (1)) have an essentially affirmative answer. We first consider finite rings.

Let R be a finite commutative ring with identity 1. If γ and δ are idempotents in R, we write $\gamma \geqslant \delta$ if $\gamma\delta = \delta$, and obtain thus the usual partial ordering of idempotents; idempotent ϵ is *minimal* if $\epsilon \neq 0$ and if $\delta \leqslant \epsilon$ implies $\delta = 0$ or $\delta = \epsilon$. Let $(\epsilon_j)_{1 \leqslant j \leqslant n}$ be the set of all minimal idempotents. Then if $j \neq k$, $\epsilon_j\epsilon_k = 0$, and clearly $1 = \sum_1^n \epsilon_j$ (for otherwise, as R is finite, idempotent $1 - \sum_1^n \epsilon_j \geqslant$ some minimal idempotent not in $(\epsilon_j)_{1 \leqslant j \leqslant n}$.) Then R is the direct sum of ideals $(R\epsilon_j)_{1 \leqslant j \leqslant n}$, and every idempotent is the sum of a subfamily of $(\epsilon_j)_{1 \leqslant j \leqslant n}$ (so there exist exactly 2^n idempotents in R). If X is a subset of A, let $\phi_X \in R^A$ be its characteristic function. If u is a character of R^A, $u(\phi_X)$ is then an idempotent in R since ϕ_X is an idempotent in R^A.

THEOREM 3. *Let R be a finite commutative ring with identity, $\epsilon_1, \ldots, \epsilon_n$ $(n \geqslant 1)$ its minimal idempotents, and let Φ be the class of all ultrafilters on A. For each character u of R^A and for $1 \leqslant j \leqslant n$, let $\mathfrak{F}_{u,j} = [X \subseteq A : u(\phi_X) \geqslant \epsilon_j]$. Then $u \rightarrow (\mathfrak{F}_{u,1}, \ldots, \mathfrak{F}_{u,n})$ is a one-to-one map from $M(R^A)$ onto Φ^n. Hence if A is finite and has m members, $M(R^A)$ has m^n members; if A is infinite with cardinality $\aleph$, $M(R^A)$ has cardinality* $\exp(\exp(\aleph))$.

Proof. $\mathfrak{F}_{u,j}$ is an ultrafilter: as $u(\phi_A) = 1 \geqslant \epsilon_j$ and $u(\phi_\phi) = 0 < \epsilon_j$, $A \in \mathfrak{F}_{u,j}$ and $\phi \notin \mathfrak{F}_{u,j}$; if $X, Y \in \mathfrak{F}_{u,j}$, $u(\phi_{X \cap Y}) = u(\phi_X)u(\phi_Y) \geqslant \epsilon_j^2 = \epsilon_j$, so $X \cap Y \in \mathfrak{F}_{u,j}$; if $X \in \mathfrak{F}_{u,j}$ and $Y \supseteq X$, $\phi_X = \phi_X\phi_Y$, so $u(\phi_X) = u(\phi_X)u(\phi_Y)$, that is, $u(\phi_Y) \geqslant u(\phi_X) \geqslant \epsilon_j$, and therefore $Y \in \mathfrak{F}_{u,j}$; finally, if $X \notin \mathfrak{F}_{u,j}$, $u(\phi_X)\epsilon_j = 0$ by minimality of ϵ_j, so

$$u(\phi_{A-X})\epsilon_j = [u(\phi_{A-X}) + u(\phi_X)]\epsilon_j = u(\phi_A)\epsilon_j = \epsilon_j,$$

that is, $u(\phi_{A-X}) \geqslant \epsilon_j$, and therefore $A - X \in \mathfrak{F}_{u,j}$. Thus $\mathfrak{F}_{u,j}$ is an ultrafilter. Next, suppose $\mathfrak{F}_{u,j} = \mathfrak{F}_{v,j}$ for $1 \leqslant j \leqslant n$. Given subset X of A,

$$\begin{aligned} u(\phi_X) = \sum[\epsilon_j : \epsilon_j \leqslant u(\phi_X)] &= \sum[\epsilon_j ; X \in \mathfrak{F}_{u,j}] = \sum[\epsilon_j : X \in \mathfrak{F}_{v,j}] \\ &= \sum[\epsilon_j : \epsilon_j \leqslant v(\phi_X)] = v(\phi_X). \end{aligned}$$

As R is finite, the functions ϕ_X generate R^A; hence $u = v$. Thus the map is one-to-one. Next, let $\mathfrak{F}_1, \ldots, \mathfrak{F}_n$ be any n (not necessarily distinct) ultrafilters on A, and let $1 \leqslant j \leqslant n$. As R is finite, for each $f \in R^A$ there exists one and only one $\lambda_{f,j} \in R$ such that $f^{-1}(\lambda_{f,j}) \in \mathfrak{F}_j$. If $f, g \in R^A$ and if $\mu \in R$, there exists

$$\alpha \in f^{-1}(\lambda_{f,j}) \cap g^{-1}(\lambda_{g,j}) \cap (f+g)^{-1}(\lambda_{f+g,j}) \cap (fg)^{-1}(\lambda_{fg,j}) \cap (\mu f)^{-1}(\lambda_{\mu f,j})$$

since $\mathfrak{F}_j$ is a filter. Hence

$$\begin{aligned} \lambda_{f+g,j} &= (f+g)(\alpha) = f(\alpha) + g(\alpha) = \lambda_{f,j} + \lambda_{g,j}, \\ \lambda_{fg,j} &= (fg)(\alpha) = f(\alpha)g(\alpha) = \lambda_{f,j}\lambda_{g,j}, \end{aligned}$$

and

$$\lambda_{\mu f,j} = (\mu f)(\alpha) = \mu f(\alpha) = \mu\lambda_{f,j}.$$

Let

$$u(f) = \sum_{j=1}^{n} \lambda_{f,j}\epsilon_j.$$

Then clearly from the above u is linear, and for any $f, g \in R^A$,

$$u(f)u(g) = \left(\sum_{j=1}^{n} \lambda_{f,j}\epsilon_j\right)\left(\sum_{k=1}^{n} \lambda_{g,k}\epsilon_k\right) = \sum_{j=1}^{n}\sum_{k=1}^{n} \lambda_{f,j}\lambda_{g,k}\epsilon_j\epsilon_k$$
$$= \sum_{j=1}^{n} \lambda_{f,j}\lambda_{g,j}\epsilon_j = \sum_{j=1}^{n} \lambda_{fg,j}\epsilon_j = u(fg).$$

Also, clearly $\lambda_{e,j} = 1$. Hence $u \in M(R^A)$. For any subset X of A,

$$1 = \lambda_{\phi_X,j}$$

if and only if $X \in \mathfrak{F}_j$, and

$$0 = \lambda_{\phi_X,j}$$

if and only if $X \notin \mathfrak{F}_j$; therefore $u(\phi_X) = \sum[\epsilon_j : X \in \mathfrak{F}_j]$, and so $u(\phi_X) \geqslant \epsilon_j$ if and only if $X \in \mathfrak{F}_j$. Hence $\mathfrak{F}_{u,j} = \mathfrak{F}_j$ for $1 \leqslant j \leqslant n$, and therefore the map is onto Φ^n. If A is finite with m members, every ultrafilter on A is a point ultrafilter, and therefore Φ^n has m^n members. Suppose A is infinite with cardinality $\aleph$. Then Φ has cardinality exp $(\exp(\aleph))$ (**2**, Exercise 14(c), p. **73**), so $M(R^A)$ has cardinality $[\exp(\exp(\aleph))]^n = \exp(\exp(\aleph))$.

We see therefore that the answer to question (2) is in general negative if R is finite:

COROLLARY. *Let R be a finite commutative ring with identity* 1, *A a set containing more than one element. Then every character of R^A is a projection if and only if A is finite and the only idempotents of R are* 0 *and* 1.

5. Algebras over integral domains.

THEOREM 4. *Let D be an integral domain, K its field of quotients. The following two conditions are both necessary and sufficient for every character of the D-algebra D^A to be a projection:*

(1) *Every character of the K-algebra K^A is a projection;*

(2) *For every $u \in M(D^A)$ and every $f \in D^A$ such that $f(\alpha) \neq 0$ for all $\alpha \in A$, $u(f) \neq 0$.*

Proof. Necessity: if u is a projection of D^A and if $f(\alpha) \neq 0$ for all $\alpha \in A$, then $u(f) \neq 0$; hence (2) is necessary. If (1) does not hold, there exist characters of K^A which are not projections. Then by Corollary 1 of Theorem 2, $\mathfrak{U}_K(A)$ is an incomplete uniform structure. Let $\mathfrak{F}$ be a non-convergent Cauchy filter on A for $\mathfrak{U}_K(A)$. Then for any $f \in D^A$, $f(\mathfrak{F})$ is a Cauchy filter base in D, hence $\lim f(\mathfrak{F})$ exists and lies in D since $D \subseteq K$ is closed and K complete. Clearly $u: f \to \lim f(\mathfrak{F})$ is a character of D^A, and for the characteristic function ϕ_α of any $\{\alpha\}$, $\alpha \in A$, $u(\phi_\alpha) = 0$ since $\mathfrak{F}$ is not convergent. Thus u is not a projection.

Sufficiency: let $u \in M(D^A)$. If $f \in K^A$, f is the quotient g/h of functions $g, h \in D^A$ where $h(\alpha) \neq 0$ for all $\alpha \in A$, and so by (2) $u(h) \neq 0$. It is easy to see that if we define $v(g/h)$ to be $u(g)/u(h)$ for all $g, h \in D^A$ such that $h(\alpha) \neq 0$ for all $\alpha \in A$, then v is a well-defined character of K^A extending u. As v is a projection by (1), u is also a projection.

In discussing principal domains we shall use the terminology and results of § 1 of Bourbaki's *Algèbre*, chapter 7. Also, we assume as part of the definition of compactness that compact spaces are separated.

THEOREM 5. *Let D be a principal domain possessing at least two non-associated extremal elements π and σ. If A either admits no Ulam measure or has cardinality not greater than that of D, every character of D^A is a projection.*

Proof. By Theorem 4 and Theorem B, it suffices to prove that if u is a character of D^A, then $u(f) \neq 0$ for every $f \in D^A$ satisfying $f(\alpha) \neq 0$ for all $\alpha \in A$. Define $p \in D^A$ such that for each $\alpha \in A$, $p(\alpha)$ is the highest power of π dividing $f(\alpha)$. Then $f = pq$ with $q \in D^A$ such that $(\pi, q(\alpha)) = 1$ for each $\alpha \in A$, and hence there exist $g_1, g_2 \in D^A$ such that $\pi g_1 + q g_2 = e$. This implies $\pi u(g_1) + u(q)u(g_2) = 1$ and therefore $u(q) \neq 0$ since π is not invertible. Similarly, there exist $h_1, h_2 \in D^A$ such that $\sigma h_1 + p h_2 = e$, yielding $u(p) \neq 0$. Therefore $u(f) = u(p)u(q) \neq 0$.

The author is indebted to the referee for the following theorem and remark.

THEOREM 6. *If R is a compact commutative ring with identity and if A is any infinite set, there exist characters of R^A which are not projections.*

Proof. Let $\mathfrak{U}$ be any ultrafilter on A which is not a point ultrafilter. For any $f \in R^A$, $f(\mathfrak{U})$ is an ultrafilter base on R and thus converges. Hence u: $f \to \lim f(\mathfrak{U})$ is clearly a character of R^A which is not a projection.

If D is the ring of p-adic integers for some prime p, D is a compact principal domain; if A is a countably infinite set, A admits no Ulam measure but there exist characters of D^A which are not projections by Theorem 6. Thus the condition in Theorem 5 that D have at least two non-associated extremal elements cannot be omitted without other restrictions on D.

6. Compact principal domains. Let K be a field with a discrete valuation v of rank 1. Then the valuation ring $D = [x \in K : v(x) \geqslant 0]$ is a principal domain whose field of quotients is K, and $P = [x \in K : v(x) > 0]$ is the unique maximal ideal of D. If the topology on K defined by v is locally compact (equivalently, if K is complete and if the residue class field D/P is finite (**3**, Exercise 24, p. 59)), D is compact.

Thus finite fields and valuation rings of locally compact fields whose topology is given by a discrete valuation of rank 1 are compact principal domains. We now show these are the only compact principal domains.

THEOREM 7. *If D is an infinite compact principal domain, then there exists a non-trivial discrete valuation v of rank 1 on the field of quotients K of D such that:*

(1) *The topology of K defined by v is locally compact and induces on D its given topology.*

(2) *D is the valuation ring of K with respect to v.*

Proof. As in the example following Theorem 6, if A is a countably infinite set, A admits no Ulam measure but by Theorem 6 there exist characters of D^A which are not projections; hence as D is infinite, D is not a field by Theorem B, so by Theorem 5 there exists an extremal element $p \in D$ such that $\{p\}$ is a representative system of extremal elements. For each non-zero element x of K there exist a unique unit u of D and a unique integer n such that $x = up^n$; if $x = up^n$, let $v(x) = n$, and let $v(0) = +\infty$. Clearly v is a discrete valuation of rank 1 on K and its valuation ring is D. The topology $\mathfrak{T}_v$ induced on D by the topology of K defined by v is separated, and the given topology $\mathfrak{T}$ of D is compact. Hence to prove $\mathfrak{T} = \mathfrak{T}_v$ it suffices to show $\mathfrak{T}_v$ is weaker than $\mathfrak{T}$, that is, for all positive integers n, $U_n = [x \in D : v(x) > n]$ is a neighbourhood of 0 for $\mathfrak{T}$. Let V be a neighbourhood of 0 for $\mathfrak{T}$ not containing $1, p, p^2, \ldots, p^n$. By (**3**, Exercise 7, p. 56) there exists a neighbourhood W of 0 for $\mathfrak{T}$ satisfying $DW \subseteq V$. Let $x = up^k \in W$. If $k \leq n$, $p^k = u^{-1}(up^k) \in DW \subseteq V$, a contradiction. Hence $k > n$, that is, $x \in U_n$. Thus $W \subseteq U_n$, so $\mathfrak{T}_v = \mathfrak{T}$. But then D is a compact neighbourhood of 0 for the topology of K, so K is locally compact.

Corollary. *A compact principal domain is metrizable, totally disconnected, and has exactly one maximal ideal.*

7. A topological application. Let T be a topological space, $\mathfrak{C}(T)$ the algebra over the real numbers $\mathbf{R}$ of all continuous real-valued functions on T. We shall apply Theorem B to give necessary and sufficient conditions on $\mathfrak{C}(T)$ for every connected component of T to be open. Let us call an algebra *decomposable* if it is the direct sum of two proper ideals, *indecomposable* otherwise. The following theorem is well known and easy to prove:

Theorem 8. *T is connected if and only if* $\mathfrak{C}(T)$ *is indecomposable.*

Let us call *Ulam's Axiom* the assertion that there exist no Ulam measures; it is known that Ulam's Axiom is consistent with the usual axioms of set theory (**9**, pp. 207–8). Let us call an algebra *fully decomposable* if it is isomorphic with the Cartesian product of indecomposable algebras.

Theorem 9. *If every connected component of T is open,* $\mathfrak{C}(T)$ *is fully decomposable. Conversely, Ulam's Axiom is equivalent to the following assertion: if T is any topological space such that* $\mathfrak{C}(T)$ *is fully decomposable, then every connected component of T is open.*

Proof. If $(T_\alpha)_{\alpha \in A}$ is the family of all connected components of T and if each T_α is open, clearly $\mathfrak{C}(T)$ is isomorphic with $\Pi_{\alpha \in A}\,\mathfrak{C}(T_\alpha)$, and by Theorem

8 each $\mathfrak{C}(T_\alpha)$ is indecomposable. To prove the second assertion, we shall first prove the following lemma:

LEMMA. *Let $(E_\alpha)_{\alpha \in A}$ be a family of non-zero algebras over the real numbers* $\mathbf{R}$ *indexed by a set A admitting no Ulam measure, and let g be an isomorphism from $E = \Pi_{\alpha \in A} E_\alpha$ onto $\mathfrak{C}(T)$. Then T is the topological sum of a family $(T_\alpha)_{\alpha \in A}$ of subsets, also indexed by A, and for all $\alpha \in A$ $\rho_\alpha \circ g \circ i_\alpha$ is an isomorphism from E_α onto $\mathfrak{C}(T_\alpha)$, where i_α is the canonical injection isomorphism from E_α into E, ρ_α the restriction homomorphism from $\mathfrak{C}(T)$ into $\mathfrak{C}(T_\alpha)$.*

Proof. E has an identity e as it is isomorphic with $\mathfrak{C}(T)$, so for all $\alpha \in A$, E_α has $e_\alpha = pr_\alpha(e)$ as its identity. Then $h_\alpha = (g \circ i_\alpha)(e_\alpha) \in \mathfrak{C}(T)$ is an idempotent, hence the characteristic function of an open-closed set $T_\alpha \subseteq T$. Now $\alpha \neq \beta$ implies $i_\alpha(e_\alpha) \cdot i_\beta(e_\beta) = 0$ and thus $h_\alpha \cdot h_\beta = 0$, that is, $T_\alpha \cap T_\beta = \phi$. Furthermore, for any $t \in T$, $t^\wedge \circ g$ is a character of E and hence, by Theorem B, $t^\wedge \circ g = v_\alpha \circ pr_\alpha$ for some $\alpha \in A$ and some character v_α of E. Then $h_\alpha(t) = t^\wedge(h_\alpha) = t^\wedge(g(i_\alpha(e_\alpha))) = (v_\alpha \circ pr_\alpha)(i_\alpha(e_\alpha)) = v_\alpha(e_\alpha) = 1$ and therefore $t \in T_\alpha$; this shows T is the union and hence the topological sum of $(T_\alpha)_{\alpha \in A}$. Next, suppose $(\rho_\beta \circ g \circ i_\beta)(x) = 0$ for some $x \in E_\beta$. Then $(g \circ i_\beta(x))(t) = (\rho_\beta \circ g \circ i_\beta(x))(t) = 0$ for $t \in T_\beta$, whereas for $t \in T_\alpha \neq T_\beta$, $h_\beta(t) = 0$ and hence

$$(g \circ i_\beta(x))(t) = (g \circ i_\beta(xe_\beta))(t) = (g \circ i_\beta(x))(t) \cdot h_\beta(t) = 0;$$

this means $g \circ i_\beta(x) = 0$ and therefore, as g and i_β are isomorphisms, $x = 0$. Hence $\rho_\beta \circ g \circ i_\beta$ is one-to-one. Finally, $\rho_\beta \circ g \circ i_\beta$ is onto: for any $f_\beta \in \mathfrak{C}(T_\beta)$ let $f \in \mathfrak{C}(T)$ be the function defined by $f(t) = f_\beta(t)$ if $t \in T_\beta$, $f(t) = 0$ otherwise. Since $f \cdot h_\alpha = 0$ for any $\alpha \neq \beta$, we have

$$\begin{aligned} pr_\alpha(g^{-1}(f)) = pr_\alpha(g^{-1}(f)) \cdot pr_\alpha(i_\alpha(e_\alpha)) &= pr_\alpha(g^{-1}(f) \cdot i_\alpha(e_\alpha)) \\ &= pr_\alpha(g^{-1}(f) \cdot g^{-1}(h_\alpha)) = (pr_\alpha \circ g^{-1})(f \cdot h_\alpha) = 0 \end{aligned}$$

for $\alpha \neq \beta$. This implies $(i_\beta \circ pr_\beta)(g^{-1}(f)) = g^{-1}(f)$ and thus

$$(\rho_\beta \circ g \circ i_\beta)(pr_\beta(g^{-1}(f))) = \rho_\beta(f) = f_\beta,$$

where $pr_\beta(g^{-1}(f)) \in E_\beta$.

We return now to the proof of the theorem. Let us assume Ulam's Axiom and suppose $\mathfrak{C}(T)$ is fully decomposable. Then by the lemma, T is the topological sum of a family $(T_\alpha)_{\alpha \in A}$ of subsets, and for all $\alpha \in A$, $\mathfrak{C}(T_\alpha)$ is isomorphic with an indecomposable algebra. Hence by Theorem 8 each T_α is connected. $(T_\alpha)_{\alpha \in A}$ is therefore the set of all connected components of T, and each T_α is open. Finally, suppose Ulam's Axiom is false. Now Ulam's Axiom is equivalent to the assertion that every discrete space S is a Q-space (that is, the weakest uniform structure $\mathfrak{B}_{\mathbf{R}}(S)$ on S for which each $f \in \mathfrak{C}(S)$ is uniformly continuous is complete). (Q-spaces are defined and discussed in **(7)**; a summary of results about Q-spaces is contained in **(6**, pp. 351–2**)**, and their relation to Ulam's Axiom is discussed in **(9**, pp. 206–8**)**.) Therefore there exists a discrete space S which is not a Q-space. Let T be the completion of S for

$\mathfrak{B}_{\mathbf{R}}(S)$. Then $\mathfrak{C}(T)$ is isomorphic with $\mathfrak{C}(S)$, and as S is discrete, $\mathfrak{C}(S) = \mathbf{R}^S$, a fully decomposable algebra. Therefore $\mathfrak{C}(T)$ is fully decomposable. For any $s \in S$, there exists an open set V in T such that $V \cap S = \{s\}$; then $s \in V = V \cap \bar{S} \subseteq (V \cap S)^- = \{s\}^- = \{s\}$, so $\{s\}$ is both open and closed in T. Let C be a connected component of T containing some point in $T - S$. Then by the preceding, $s \notin C$ for all $s \in S$. Hence $C \subseteq T - S$. But then C cannot be open, since S is dense in T. $\mathfrak{C}(T)$ is therefore fully decomposable, but not every connected component of T is open.

COROLLARY. *Assume Ulam's Axiom. A topological space T is locally connected if and only if for every open subset G of T, $\mathfrak{C}(G)$ is fully decomposable.*

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F-RINGS OF CONTINUOUS FUNCTIONS I

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1. Introduction. It is well known **(2, 4)** that the ring of all real (complex) continuous functions on a compact Hausdorff space can be characterized algebraically as a Banach algebra which satisfies certain additional intrinsic conditions. It might be expected that rings of all continuous functions on other topological spaces also have algebraic characterizations. The main purpose of this note is to discuss two such characterizations. In both cases the characterizations are given in the terms of the theory of F-rings **(1)**. In one case a characterization is given for the ring of all (real) continuous functions on a ***generalized P-space***, that is, a zero-dimensional topological space in which the class of open-closed sets forms a σ-algebra. A Hausdorff generalized P-space is a P-space in the terminology of **(3)**. In the other case a theorem of Sikorski **(6)** is employed to give a characterization of the ring of all (real) continuous functions on an upper $\aleph_1$-compact P-space. A P-space is said to be upper $\aleph_1$-compact if every open covering of the space can be replaced by an at most countable subcovering.

It is remarked in a previous paper **(1)** that the ring of all (real) continuous functions on a P-space is a regular M-ring and hence is the ring of all $(\Omega, \mathfrak{F})$-measurable functions where Ω is the domain of the continuous function ring and $\mathfrak{F}$ is a certain distinguished σ-algebra of subsets of Ω. A portion of this note is devoted to a characterization of those $\mathfrak{F}$'s for which a topology $\mathfrak{T}$ exists such that the ring of all $(\Omega, \mathfrak{F})$-measurable functions is exactly the ring of all real functions on Ω which are continuous under the topology $\mathfrak{T}$.

The notation and definitions of **(1)** are used here with the exception that $M(\Omega, \mathfrak{F})$ is used to denote the regular M-ring of all $(\Omega, \mathfrak{F})$-measurable functions. Since the following discussion involves a multiplicity of topologies defined on a given abstract space, the notation $[\Omega, \mathfrak{T}]$ is used to designate a topological space having Ω as its set of points and $\mathfrak{T}$ as its class of open sets. $C[\Omega, \mathfrak{T}]$ designates the ring of all (real) continuous functions on $[\Omega, \mathfrak{T}]$. When there is no ambiguity the shortened forms $M(\mathfrak{F})$ and $C[\mathfrak{T}]$ are used respectively for $M(\Omega, \mathfrak{F})$ and $C[\Omega, \mathfrak{T}]$.

If $\mathfrak{A}$ is an algebra of subsets of some set Ω, then $T\mathfrak{A}$ stands for the class of open sets in the topology generated by $\mathfrak{A}$, and if $[\Omega, \mathfrak{T}]$ is a topological space, $\zeta\mathfrak{T}$ stands for the class of all open-closed sets in $[\Omega, \mathfrak{T}]$. If $p \in \Omega$, then $\mathfrak{M}_p$ designates the set $\{A \in \mathfrak{A} \mid p \notin A\}$ which is clearly a maximal ideal of $\mathfrak{A}$.

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If $\mathfrak{D}$ is a collection of sets, then $\bigcap \mathfrak{D}$ and $\bigcup \mathfrak{D}$ designate respectively the intersection and union of the elements in $\mathfrak{D}$.

2. Generalized P-spaces and σ-algebras. Let Ω be a set which contains at least two points and let $\mathfrak{F}$ be a σ-algebra of subsets of Ω. If $\mathfrak{T}$ is a topology on Ω such that $M(\mathfrak{F})$ is the ring of all continuous functions on $[\Omega, \mathfrak{T}]$, then $\mathfrak{F} \subseteq \mathfrak{T}$ and hence $T\mathfrak{F} \subseteq \mathfrak{T}$. It is a matter of direct verification to show that in general $[\Omega, T\mathfrak{F}]$ is a generalized P-space and $M(\mathfrak{F}) \subseteq C[T\mathfrak{F}]$. Thus if there is a $\mathfrak{T}$ such that $M(\mathfrak{F}) = C[\mathfrak{T}]$, then $M(\mathfrak{F}) = C[T\mathfrak{F}] = C[\mathfrak{T}]$. Hence the problem of determining those $\mathfrak{F}$ for which there exists a $\mathfrak{T}$ such that $C[\mathfrak{T}] = M(\mathfrak{F})$ is reduced to the problem of determining those $\mathfrak{F}$ for which $C[T\mathfrak{F}] = M(\mathfrak{F})$.

THEOREM 1. *If $\mathfrak{F}$ is a σ-algebra of subsets of Ω, then the following statements are equivalent:*

(i) $C[T\mathfrak{F}] = M(\mathfrak{F})$,

(ii) $\zeta T\mathfrak{F} = \mathfrak{F}$,

(iii) *If $\mathfrak{B}$ is an ideal of $\mathfrak{F}$ with properties:*

(a) $\mathfrak{B} = \bigcap \{\mathfrak{M}_p \mid \mathfrak{M}_p \supseteq \mathfrak{B}\}$,

(b) $\mathfrak{M}_p \supseteq \mathfrak{B} \Rightarrow$ *there is a principal ideal $\mathfrak{J}_p$ such that $\mathfrak{B} \subseteq \mathfrak{J}_p \subseteq \mathfrak{M}_p$, then $\mathfrak{B}$ is principal.*

Proof. (i) $\Rightarrow$ (ii): If A is an open-closed set in the topology $T\mathfrak{F}$, then χ_A is continuous and hence belongs to $C[T\mathfrak{F}] = M(\mathfrak{F})$. Thus it is clear that $A = \{p \in \Omega \mid \chi_A > \frac{1}{2}\}$ belongs to $\mathfrak{F}$. Therefore $\zeta T\mathfrak{F} \subseteq \mathfrak{F}$. Obviously $\mathfrak{F} \subseteq \zeta T\mathfrak{F}$.

(ii) $\Rightarrow$ (iii): If $\mathfrak{B}$ has property (iiia), then $\mathfrak{B} = \{A \in \mathfrak{F} \mid A \subseteq D\}$ where $D = \bigcup\mathfrak{B}$. Indeed, suppose $A \subseteq D$. Then if $\mathfrak{M}_p \supseteq \mathfrak{B}$, it follows that $p \notin D$, and therefore $\mathfrak{M}_p \supseteq \mathfrak{B}$ implies that $A \in \mathfrak{M}_p$. Thus $A \in \bigcap \{\mathfrak{M}_p \mid \mathfrak{M}_p \supseteq \mathfrak{B}\} = \mathfrak{B}$.

If $\mathfrak{B}$ satisfies both (iiia) and (iiib), then $\mathfrak{B}$ is principal. This is proved by showing that D is open-closed and then noting that $D \in \mathfrak{B}$. Let $p \notin D$. Then $\mathfrak{M}_p \supseteq \mathfrak{B}$ and there is a principal ideal $\mathfrak{J}_p$ such that $\mathfrak{B} \subseteq \mathfrak{J}_p \subseteq \mathfrak{M}_p$. If I_p is the generator of $\mathfrak{J}_p$, then $p \notin I_p$ and $B \subseteq I_p$ for all $B \in \mathfrak{B}$. Thus $D \subseteq I_p$ and $p \in \Omega - I_p \in \mathfrak{F}$. Therefore D is open-closed and belongs to $\mathfrak{F}$. If $D \notin \mathfrak{B}$, then for some $\mathfrak{M}_p \supseteq \mathfrak{B}$ we have $p \in D$ which leads to a contradiction because $\mathfrak{M}_p \supseteq \mathfrak{B}$ implies $p \notin D$. This proves that $\mathfrak{B}$ is the principal ideal generated by D.

(iii) $\Rightarrow$ (ii): Let D be an open-closed set in the topology $T\mathfrak{F}$. The ideal $\mathfrak{D} = \{A \in \mathfrak{F} \mid A \subseteq D\}$ satisfies (iiia). Indeed, if $A \in \bigcap \{\mathfrak{M}_p \mid \mathfrak{M}_p \supseteq \mathfrak{D}\}$, then $p \notin A$ for all $\mathfrak{M}_p \supseteq \mathfrak{D}$. Since $\mathfrak{M}_p \supseteq \mathfrak{D}$ is equivalent to $p \notin D$, it then follows that $A \in \bigcap \{\mathfrak{M}_p \mid \mathfrak{M}_p \supseteq \mathfrak{D}\}$ implies $p \notin A$ for all $p \notin D$ which in turn implies $A \subseteq D$. Thus $\bigcap \{\mathfrak{M}_p \mid \mathfrak{M}_p \supseteq \mathfrak{D}\} = \mathfrak{D}$.

If $p \notin D$, then there is a $B \in \mathfrak{F}$ such that $p \in B$ and $B \cap D = \phi$, because D is closed. Hence $\Omega - B \supseteq D$ and the principal ideal $\mathfrak{J}_p$ generated by $\Omega - B$ contains $\mathfrak{D}$. Therefore if $\mathfrak{M}_p \supseteq \mathfrak{D}$, there is a principal ideal $\mathfrak{J}_p$ such that $\mathfrak{D} \subseteq \mathfrak{J}_p \subseteq \mathfrak{M}_p$. Thus $\mathfrak{D}$ satisfies (iiib) and $\mathfrak{D}$ is principal.

Now $\cup \mathfrak{D} = D$, because D is open. Since $\mathfrak{D}$ is principal, D is its generator; hence $D \in \mathfrak{F}$ and $\mathfrak{F} = \zeta T\mathfrak{F}$.

(ii) $\Rightarrow$ (i): Since $M(\mathfrak{F}) \subseteq C[T\mathfrak{F}]$, statement (i) is established if we show that $C[T\mathfrak{F}] \subseteq M(\mathfrak{F})$. Suppose $\mathfrak{g} \in C[T\mathfrak{F}]$. Then as in **(3**, Theorem 5.3**)**, $\mathfrak{g}^{-1}(\alpha) = \{p \in \Omega \mid \mathfrak{g}(p) = \alpha\}$ is an open set for any real α (because countable intersections of sets in $T\mathfrak{F}$ belong to $T\mathfrak{F}$) and hence

$$\{p \in \Omega \mid \mathfrak{g}(p) < \lambda\} = \cup \{\mathfrak{g}^{-1}(\alpha) \mid \alpha < \lambda\} \in \zeta T\mathfrak{F} = \mathfrak{F}.$$

Therefore $\mathfrak{g} \in M(\mathfrak{F})$.

As an example of an $\mathfrak{F}$ for which $C[T\mathfrak{F}] \neq M(\mathfrak{F})$, let $\mathfrak{F}$ be the class of all Borel sets on the real line E. It is clear that $T\mathfrak{F}$ is the discrete topology on E and that $C[T\mathfrak{F}]$ is the ring of all real functions on E.

In Theorem 1 the proof that (ii) $\Leftrightarrow$ (iii) does not depend on the hypothesis that $\mathfrak{F}$ is an σ-algebra. Therefore the following more general result can be obtained:

COROLLARY 1. *Let $\mathfrak{A}$ be an algebra of subsets of Ω. Then $\mathfrak{A}$ satisfies condition* (ii) *if and only if* $(\Omega, \mathfrak{A})$ *satisfies condition* (iii).

The following corollary, useful in the sequel, is a direct consequence of Theorem 1:

COROLLARY 2. *If* $[\Omega, \mathfrak{T}]$ *is a generalized P-space, then* $C[\Omega, \mathfrak{T}] = M(\Omega, \zeta\mathfrak{T})$.

Indeed, in a generalized P-space, $T\zeta\mathfrak{T} = \mathfrak{T}$ and $\zeta T\zeta\mathfrak{T} = \zeta\mathfrak{T}$. Hence by Theorem 1 the equation $C[\Omega, \mathfrak{T}] = M(\Omega, \zeta\mathfrak{T})$ is valid.

3. A maximal ideal theorem. Before consideration of the characterizations, a theorem and a corollary concerning maximal ideals are proved. These make clear the relationship between the concepts of closed maximal ideal and real maximal ideal for regular F-rings.

THEOREM 3. *Let R be a regular F-ring. If a maximal ideal $\mathfrak{M}$ of R is real, it is also closed.*

For definitions of real and closed see **(1)**. In order to prove Theorem 3 the following lemma is employed.

LEMMA 1. *If $\{e_n\}$ is an orthogonal sequence of idempotents of R such that* $\bigvee_{n=1}^{\infty} e_n = 1$ *and if $\{\alpha_n\}$ is a sequence of real numbers such that* $0 < \epsilon < \alpha_n$ *for all integers* $n > 1$, *then* $\bigvee_{n=1}^{\infty} \alpha_n e_n \in R$.

Proof. The element $b = \bigvee_{n=1}^{\infty} 1/\alpha_n\, e_n$ belongs to R. From the equation

$$\bar{e}_b = \bigvee_{m=1}^{\infty} (1 \wedge mb) = \bigvee_{m=1}^{\infty} [1 \wedge m\, (\bigvee_{n=1}^{\infty} e_n/\alpha_n],$$

we deduce by the distributivity of the lattice operations that $\bar{e}_b = \bigvee_{n=1}^{\infty} e_n = 1$. Hence the regularity of R implies **(1**, Theorem 2**)** that $b^{-1} \in R$. Clearly

$$e_m = b\, b^{-1}\, e_m = (\bigvee_{n=1}^{\infty} e_n/\alpha_n)\, b^{-1}\, e_m = e_n/\alpha_n\, b^{-1},$$

so $\alpha_n e_n = b^{-1}e$ for $n \geqslant 1$. Therefore, since $b^{-1} = (\mathrm{V}_{n=1}{}^{\infty} e_n)\, b^{-1} = \mathrm{V}_{n=1}{}^{\infty} (e_n b^{-1})$, it follows that $b^{-1} = \mathrm{V}_{n=1}{}^{\infty} \alpha_n e_n$.

Proof of Theorem 3. Suppose $\mathfrak{M}$ is not closed. Then there is a sequence $\{e_n\}$ of orthogonal idempotents such that $\mathrm{V}_{n=1}{}^{\infty} e_n = 1$ and $e_n \in \mathfrak{M}$ for each integer $n \geqslant 1$. Indeed, there is a sequence $\{b_n\}$ of elements of $\mathfrak{M}$, such that $b_n \geqslant 0$ for all integers $n \geqslant 1$ and $b = \mathrm{V}_{n=1}{}^{\infty} b_n \in R$ while $b \notin \mathfrak{M}$. Since

$$\mathrm{V}_{n=1}^{\infty}\, \bar{e}_{b_n} = \mathrm{V}_{n=1}^{\infty}\, \mathrm{V}_{m=1}^{\infty}\, [1 \wedge m b_n] = \mathrm{V}_{m=1}^{\infty}\, [1 \wedge m\, \mathrm{V}_{n=1}^{\infty}\, b_n] = \bar{e}_b$$

and since $x \in \mathfrak{M}$ if and only if $\bar{e}_x \in \mathfrak{M}$, it follows that $\bar{e}_b$ does not belong to $\mathfrak{M}$. Therefore $e_0 = 1 - \bar{e}_b \in \mathfrak{M}$. Let

$$e_n = [e_0 \vee (\mathrm{V}_{m=1}^{n}\, \bar{e}_{b_m})] \wedge \{1 - [e_0 \vee (\mathrm{V}_{m=1}^{n-1}\, \bar{e}_{b_m})]\}$$

for $n \geqslant 1$. Then it can be shown that $\{e_0, e_1, \ldots\}$ is an orthogonal sequence of idempotents such that $\mathrm{V}_{n=0}{}^{\infty} e_n = 1$, while $e_n \in \mathfrak{M}$ for each integer $n \geqslant 0$.

If $\{e_n\}$ is a sequence of orthogonal idempotents in $\mathfrak{M}$ such that $\mathrm{V}_{n=0}{}^{\infty} e_n = 1$, then by Lemma 1, $d = \mathrm{V}_{n=0}{}^{\infty} (n + 1)\, e_n$ belongs to R and has an inverse $d^{-1} \in R$. If m is a natural number, then

$$d - m \cdot 1 = \sum_{n=0}^{m-1} (n + 1 - m)\, e_n + \mathrm{V}_{n=m}^{\infty}(n + 1 - m)e_n$$

belongs to R. Since $\sum_{n=0}{}^{m-1}(n + 1 - m)e_n$ belongs to $\mathfrak{M}$ for all $m \geqslant 1$, it follows that the non-negative element $\mathrm{V}_{n=m}{}^{\infty}(n + 1 - m)e_n$ and the element $d - m \cdot 1$ have the same image under the natural homomorphism of R onto $R - \mathfrak{M}$. This homomorphism preserves order, and hence $d(\mathfrak{M}) - m \cdot 1 \geqslant 0$ for all choices of $m \geqslant 1$. Therefore $R - \mathfrak{M}$ cannot be the real field.

Corollary. *If R is a regular F-ring, then a maximal ideal is real if and only if it is closed.*

Proof. This follows from Theorem 3 and (**1**, Theorem 5).

4. Characterizations. In this section, when R is a regular F-ring $B(R)$ is used to designate the collection of idempotents of R. It is shown in (**1**) that $B(R)$ is a σ-complete Boolean algebra with respect to the lattice operations of R.

Let G be either a regular F-ring or a σ-complete Boolean algebra, and let Ω stand for the class of all closed maximal ideals of G. Consider the following conditions:

(α) If $\mathfrak{B}$ is an ideal of G of the form $\mathfrak{B} = \bigcap \{\mathfrak{M} \in \Omega \mid \mathfrak{M} \supseteq \mathfrak{B}\}$ and if for each $\mathfrak{M} \in \Omega$ such that $\mathfrak{M} \supseteq \mathfrak{B}$ there exists a principal ideal $\mathfrak{J}$ which satisfies the relation $\mathfrak{B} \subseteq \mathfrak{J} \subseteq \mathfrak{M}$, then $\mathfrak{B}$ is a principal ideal.

(β) Every proper closed ideal of G is a subset of some closed maximal ideal of G.

Lemma 2. *Let R be a regular F-ring. The ideal lattices of R and $B(R)$ are isomorphic (under $J \to J \cap B(R)$ and its inverse $J \to RJ$) such that principal ideals correspond to principal ideals and closed ideals to closed ideals.*

Proof. The isomorphism of the ideal lattices and the result that principal ideals correspond to principal ideals are trivial consequences of **(5**, Theorem 5**)**. It remains only to prove that closed ideals correspond to closed ideals. An ideal Q of R is closed if and only if for a sequence $\{f_n\}$ of non-negative elements of Q, the existence of $f = \mathrm{V}_{n=1}^{\infty} f_n$ in R implies $f \in Q$. Suppose J is a closed ideal of $B(R)$. If $\{f_n\}$ is a sequence of non-negative elements of RJ, then

$$\bar{e}_{f_n} \in J, \qquad n \geq 1,$$

and hence

$$\mathrm{V}_{n=1}^{\infty} \bar{e}_{f_n} \in J.$$

If $f = \mathrm{V}_{n=1}^{\infty} f_n$ exists in R, then as in the proof of Theorem 3 it can be shown that

$$\bar{e}_f = \mathrm{V}_{n=1}^{\infty} \bar{e}_{f_n}.$$

Hence $\bar{e}_f \in J$ and $f = f\bar{e}_f \in RJ$. Therefore RJ is also closed. If Q is a closed ideal of R, then trivially $Q \cap B(R)$ is a closed ideal of $B(R)$.

The following useful corollary is an immediate consequence of Lemma 2.

Corollary. *If R is a regular F-ring, then R satisfies condition $\alpha(\beta)$ if and only if $B(R)$ satisfies condition $\alpha(\beta)$.*

It is now possible to characterize the ring of all continuous (real) functions on a generalized P-space.

Theorem 4. *A regular F-ring R is isomorphic to the ring of all continuous functions on a generalized P-space if and only if R is an M-ring which satisfies condition α.*

Proof. Let R be an M-ring which satisfies condition α. From Theorems 7 and 8 of **(3)** it follows that there is a correspondence $\phi: x \to \hat{x}$ which maps x onto a real function $\hat{x}$ defined on Ω. At each $\mathfrak{M} \in \Omega$ the value of $\hat{x}$ is the image of x under the homomorphism with kernel $\mathfrak{M}$. This correspondence ϕ is an isomorphism of R onto an M-ring $M(\Omega, \mathfrak{F})$, and under it the Boolean algebra $B(R)$ is mapped isomorphically onto the set of characteristic functions of elements in the σ-algebra $\mathfrak{F}$. In addition $B(R) \cong \mathfrak{F}$ under the correspondence $x \to \{\mathfrak{M} \in \Omega \mid \hat{x}(\mathfrak{M}) = 1\}$, $[\Omega, T\mathfrak{F}]$ is a generalized P-space (in particular since $[\Omega, T\mathfrak{F}]$ is Hausdorff it is a P-space), and finally the ideal lattices of R and $B(R)$ are isomorphic (Lemma 2). Therefore if R satisfies α, then $\mathfrak{F}$ satisfies α also and from Theorem 1 it follows that $M(\Omega, \mathfrak{F}) = C[\Omega, T\mathfrak{F}]$.

Conversely, if $[\Omega^*, \mathfrak{T}]$ is a generalized P-space, then (Corollary 2, Theorem 1) $C[\Omega^*, \mathfrak{T}] = M(\Omega^*, \zeta\mathfrak{T})$. Therefore $C[\Omega^*, \mathfrak{T}]$ is an M-ring and the σ-algebra $\zeta\mathfrak{T}$ satisfies condition (iii) of Theorem 1. Suppose $\mathfrak{B}$ is an ideal of $\zeta\mathfrak{T}$ which satisfies the hypothesis of condition α. Let $\mathfrak{B}^* = \{\mathfrak{M}_p \mid p \in \Omega^*, \mathfrak{M}_p \supseteq \mathfrak{B}\}$. Clearly $\mathfrak{B} \subseteq \mathfrak{B}^*$ and hence $B = \cup \mathfrak{B} \subseteq B^* = \cup \mathfrak{B}^*$. Since $p \notin B$ if and only if $\mathfrak{M}_p \supseteq \mathfrak{B}$ and $p \notin B^*$ if and only if $\mathfrak{M}_p \supseteq \mathfrak{B}^*$, it follows that $B = B^*$. Suppose $A \in \mathfrak{B}^* - \mathfrak{B}$. Then there is a closed maximal ideal $\mathfrak{M}$ such that $A \notin \mathfrak{M}$

and $\mathfrak{M} \supseteq \mathfrak{B}$. There is a principal ideal $\mathfrak{J}$ with generator I such that $\mathfrak{M} \supseteq \mathfrak{J} \supseteq \mathfrak{B}$. Hence $I \supseteq B \supseteq A$. This contradicts the statement $A \notin \mathfrak{M}$; therefore $\mathfrak{B}^* = \mathfrak{B}$. From Theorem 1 it follows that $\mathfrak{B}$ is principal and hence $\zeta\mathfrak{T}$ satisfies condition α. Therefore both $B(C[\Omega^*, \mathfrak{T}])$ and $C[\Omega^*, \mathfrak{T}]$ satisfy condition α.

For the characterization of the ring of all continuous functions on an upper $\aleph_1$-compact P-space the following theorem is employed.

THEOREM 5 (Sikorski). *If Ω is the set of all closed maximal ideals of a σ-complete Boolean algebra B, then the following statements are equivalent:*

(i) *There exists a topology $\mathfrak{T}$ on Ω such that the correspondence*

$$a \rightarrow \{\mathfrak{M} \in \Omega \mid a \notin \mathfrak{M}\}$$

is an isomorphism of B onto the σ-algebra of open-closed subsets of the upper $\aleph_1$-compact P-space $[\Omega, \mathfrak{T}]$.

(ii) *B satisfies condition β.*

This theorem is a restatement of (**6**, Theorem xviii) in terms of the symbolism of the present paper.

The following theorem constitutes the characterization.

THEOREM 6. *Let R be a regular F-ring. A necessary and sufficient condition for R to be isomorphic to $C[\Omega, \mathfrak{T}]$ for some upper $\aleph_1$-compact P-space $[\Omega, \mathfrak{T}]$ is that R satisfy condition β.*

Proof. Assume R satisfies β. Since principal ideals of R are closed, every principal ideal is a subset of a closed maximal ideal. Therefore R is an M-ring. In addition (see proof of Theorem 4 and (**3**, Theorems 7 and 8)), R is isomorphic to $M(\Omega, \mathfrak{F})$ where the isomorphism maps $B(R)$ onto the σ-algebra of characteristic functions on $\mathfrak{F}$ and $\mathfrak{F} \cong B(R)$. Therefore (Lemma 2) $\mathfrak{F}$ satisfies β and since closed maximal ideals of $\mathfrak{F}$ are of the form $\mathfrak{M}_p$ for $p \in \Omega$ we deduce from Theorem 5 that $\mathfrak{F}$ is the class of open-closed sets for an upper $\aleph_1$-compact P-topology $\mathfrak{T}$ on Ω. Thus $\mathfrak{F} = \zeta\mathfrak{T}$ and from Corollary 2, Theorem 1 it follows that $C[\Omega, \mathfrak{T}] = M(\Omega, \mathfrak{F}) \cong R$.

Assume $[\Omega^*, \mathfrak{T}]$ is an upper $\aleph_1$-compact P-space. The upper $\aleph_1$-compactness implies that each closed maximal ideal of $\zeta\mathfrak{T}$ is fixed. Therefore (Theorem 5) the σ-algebra $\zeta\mathfrak{T}$ satisfies condition β; hence $M(\Omega, \zeta\mathfrak{T})$ satisfies β as well. Finally by Corollary 2, Theorem 1 we have $C[\Omega, \mathfrak{T}] = M(\Omega, \zeta\mathfrak{T})$.

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ON CERTAIN EXTENSIONS OF FUNCTION RINGS

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1. Introduction. The present note is concerned with the existence and properties of certain types of extensions of Banach algebras which allow a faithful representation as the normed ring $\mathbf{C}(E)$ of all bounded continuous real functions on some topological space E. These Banach algebras can be characterized intrinsically in various ways **(1)**; they will be called function rings here. A function ring $\mathbf{E}$ will be called a normal extension of a function ring $\mathbf{C}$ if $\mathbf{E}$ is directly indecomposable, contains $\mathbf{C}$ as a Banach subalgebra and possesses a group G of automorphisms for which $\mathbf{C}$ is the ring of invariants, that is, the set of all elements fixed under G. G will then be called a group of automorphisms of $\mathbf{E}$ over $\mathbf{C}$. If $\mathbf{E}$ is a normal extension of $\mathbf{C}$ with precisely one group of automorphisms over $\mathbf{C}$, which is then the invariance group of $\mathbf{C}$ in $\mathbf{E}$, then $\mathbf{E}$ will be called a Galois extension of $\mathbf{C}$. Such an extension will be called finite if its group is finite.

The discussions below prove, for any directly indecomposable function ring, the existence of normal extensions with arbitrarily prescribed group and give a characterization of all finite Galois extensions of a function ring $\mathbf{C} = \mathbf{C}(E)$, E completely regular, which fully decompose all maximal ideals in C, in terms of regular quasi-covering spaces of E. In the special case, for instance, where E is normal and the union of finitely many simply connected open sets, the result is that any extension $\mathbf{E}$ of the considered type is a $\mathbf{C}(X)$ given by a regular covering space (X, ϕ, E) whose Poincaré group induces isomorphically the group of $\mathbf{E}$ over $\mathbf{C}$.

The proofs for these statements are obtained through arguments relating to fibre spaces which will then be interpreted for function rings via the well-known relations of the automorphisms and subrings of such rings to the underlying spaces or their Stone-Čech extensions. As it seems advisable, for the sake of clarity, to treat these two different aspects of this subject quite separately, all the topological material needed here will be presented first, whereas the transition to function rings will be left to the latter part of the paper. This transition will be based on the following familiar facts:

(1) Any function ring $\mathbf{C}$ is the $\mathbf{C}(S)$ of a unique compact S, its maximal ideal space, and the automorphisms of $\mathbf{C}$ are all induced by space automorphisms of S.

(2) If $\mathbf{C} = \mathbf{C}(E)$ with non-compact completely regular E, then $S = \beta E$, the Stone-Čech compactification of E.

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(3) If the directly indecomposable function ring **E** is an extension of **C** and X and E are the respective maximal ideal spaces, then X is mapped continuously onto E by $\phi: \mathfrak{M} \to \mathfrak{M} \cap \mathbf{C}$, $\mathfrak{M}$ the maximal ideals of **F**.

2. Automorphisms of fibre spaces. A fibre space is a triple (X, ϕ, E) consisting of two spaces X and E together with a continuous mapping $\phi: X \to E$. Although usually not required, $\phi X = E$ will always be assumed here. The base of (X, ϕ, E) is E, the fibre above $x \in E$ is the set $\phi^{-1}x$, and the group $G(\phi)$ of (X, ϕ, E) is the group of all space automorphisms of X which transform each fibre into itself.

If X is any space and s an automorphism of X, then s induces an automorphism $\bar{s}$ in $\mathbf{C}(X)$ by means of the formula $(\bar{s}f)x = f(s^{-1}x)$. Thus, in a fibre space (X, ϕ, E) the group $G(\phi)$ induces a group $\bar{G}(\phi)$ of automorphisms in $\mathbf{C}(X)$; $s \to \bar{s}$ is a homomorphism and may, but need not be an isomorphism.

For any fibre space (X, ϕ, E) the functions $f \in \mathbf{C}(E)$ determine functions $f^* = f\phi$ on X. By a known theorem (**4**, ch. I, §9) these are precisely those $g \in \mathbf{C}(X)$ which are constant on each fibre. The transition $f \to f^*$ imbeds isomorphically the function ring $\mathbf{C}(E)$ into the function ring $\mathbf{C}(X)$; the imbedded ring will be denoted by $\mathbf{C}(E)^*$.

The first topological fact needed later on concerns the existence of certain fibre spaces. It will be assumed that any space occurring contains more than one point.

Lemma 1. *If E is a connected completely regular space and G any given group, then there exists a fibre space (X, ϕ, E) with connected completely regular X such that G is (isomorphic to) a subgroup of $G(\phi)$ acting transitively on each fibre.*

Proof. Regard G as a discrete space. Then the product space $Y = G \times E$ has G as a group of automorphisms if the action of $s \in G$ be defined by $s(t, x) = (st, x)$. Now, take a fixed $a \in E$, identify all $(s, a) \in Y$, call the resulting quotient space X and let θ be the natural mapping $Y \to X$. Since E is connected, the sets $E_s = \{s\} \times E \subseteq Y$ and $\theta E_s \subseteq X$ are also connected, and therefore $X = \bigcup \theta E_s$ is connected because $\bigcap \theta E_s$ is non-void.

The continuous mapping $(s, x) \to x$ induces a continuous mapping $\phi: X \to E$ with $\phi X = E$. Also, any $s \in G$ induces a mapping s_* of X onto itself by $s_*\theta(t, x) = \theta(st, x)$, continuous because $(t, x) \to \theta(st, x)$ is a continuous mapping of Y onto X, constant on the set of all (t, a), $t \in G$. Furthermore, since $(st)_* = s_*t_*$, any s_* has a continuous inverse and is thus an automorphism of X. Finally, the mapping $s \to s_*$ is an isomorphism since $s_*\theta(t, x) = \theta(st, x) \neq \theta(t, x)$ for all $x \neq a$ if s is not the unit transformation. Obviously, these s_* map each fibre of (X, ϕ, E) onto itself and their totality acts transitively on each fibre.

It remains to prove that X is completely regular. For this it is sufficient to show that it can be imbedded in some compact space. Let K be a compact space containing Y, which exists by the complete regularity of E, and identify

in K all the points of the closure of the set of all (s, a), $s \in G$. The quotient space L thus obtained is again compact and contains X as a subspace (**4**, ch. I, §9). This completes the proof.

Remark. Obviously, any permutation p of G gives rise to an automorphism of Y and hence to an automorphism p_* of X. If $E - \{a\}$ is connected, then the group $G(\phi)$ of (X, ϕ, E) consists precisely of these p_* since the set of all points in θE_s other than $\theta(s, a)$ which an element of $G(\phi)$ maps into the same θE_t is open-closed in $\theta E_s - \{\theta(s, a)\}$.

Lemma 1 will be employed in connection with the following statements about a fibre space (X, ϕ, E) and its group $G = G(\phi)$.

LEMMA 2. *If* $\mathbf{C}(E)$ *separates the points of* E, *then any automorphism* s *of* X *whose* $\bar{s}$ *leaves each* $g \in \mathbf{C}(E)^*$ *fixed belongs to* G. *If* $\mathbf{C}(X)$ *separates the points of* X, *then* $\bar{G}$ *is isomorphic to* G *under* $s \rightarrow \bar{s}$. *If* $H \subseteq G$ *acts transitively on each fibre, then* $\mathbf{C}(E)^*$ *is the ring of invariants of* $\bar{H}$. *If* $H \subseteq G$ *does not act transitively on each fibre and if the functions on the orbit space* X/H *separate points, then the ring of invariants of* $\bar{H}$ *is greater than* $\mathbf{C}(E)^*$.

Proof. Suppose s is an automorphism of X not belonging to G, that is, $\phi a \neq \phi(s^{-1}a)$ for some $a \in X$. Then, if $f \in \mathbf{C}(E)$ separates these two points one obviously has $\bar{s}f^* \neq f^*$ and thus $\bar{s}$ does not leave all $g \in \mathbf{C}(E)^*$ fixed. Next, consider, for any $s \in G$ which is not the unit transformation, an $a \in X$ such that $s^{-1}a \neq a$ which exists since G acts effectively by definition. Then, if $f \in \mathbf{C}(X)$ separates $s^{-1}a$ from a one has $\bar{s}f \neq f$ and thus $\bar{s}$ is not the unit transformation, either. Further, if $f \in \mathbf{C}(X)$ satisfies $\bar{s}f = f$ for all $s \in H$ where H acts transitively on each fibre, then f is constant on each fibre and hence belongs to $\mathbf{C}(E)^*$. Finally, if H does not act transitively on each fibre one has $X/H \neq E$ and there exist distinct points a and b in X/H which have the same image c under the mapping $X/H \rightarrow E$ induced by ϕ. A function $g \in \mathbf{C}(X/H)$ separating a from b then gives a $g^* \in \mathbf{C}(X)$ with $\bar{s}g^* = g^*$ for all $s \in H$ which is, however, not constant on the fibre above c.

3. Quasi-covering spaces. By a quasi-covering space is meant a fibre space (X, ϕ, E) with the property that each $x \in E$ has an open neighbourhood V whose $\phi^{-1}V$ is the union of disjoint open V' on which $\phi|V'$, the restriction of ϕ to V', is a homeomorphism onto V. An open set V with this property will be called evenly covered by (X, ϕ, E). To emphasize the role of E, (X, ϕ, E) will also be called a quasi-covering space of E. If E should be locally connected and X connected, this reduces to the usual definition of a covering space (**3**, ch. II, §VI).

A quasi-covering space (X, ϕ, E) will be called regular if its group $G(\phi)$ acts transitively on each fibre. Then, if X is connected, any $s \in G(\phi)$ other than the unit transformation has no fixed points. For if V' is the part of $\phi^{-1}V$ belonging to $x' \in \phi^{-1}x$, V assumed to be an evenly covered open neigh-

bourhood of x, and $sx' = x'$, then $sV' \cap V'$ is a neighbourhood of x' consisting of points fixed under s, whereas if $sx' = x'' \neq x'$ and V'' belongs to x'', then $s^{-1}(sV' \cap V'')$ is a neighbourhood of x' consisting of points moved by s; hence the sets of the two different kinds of points are both open and if X is connected, one of them must be void and the other X.

From the special type of Galois extensions of function rings to be considered below there arise fibre spaces (X, ϕ, E) with completely regular X and E and finite $G = G(\phi)$ such that (i) each fibre consists of as many points as G has elements and (ii) $\bar{G}$ has $\mathbf{C}(E)^*$ as its ring of invariants. In this situation one has

LEMMA 3. *(X, ϕ, E) is a regular quasi-covering space.*

Proof. First, the orbit space X/G is again completely regular. This follows from the fact that X/G is a subspace of the compact $\beta X/G'$, G' the continuous extension of G to βX, which in turn is a consequence of a known theorem on quotient spaces (**4**, ch. I, §9). Now, the functions on X/G certainly separate points, and as the ring of invariants of $\bar{G}$ is $\mathbf{C}(E)^*$, Lemma 2 implies that G acts transitively on each fibre. An immediate consequence of this and the hypothesis on the number of points in each fibre is that no $s \in G$ other than the unit has any fixed points. Further it follows that the mapping ϕ is open since $E = X/G$ and for any open $O \subseteq X$ $O^* = \bigcup sO$, $s \in G$, is again open.

Now, suppose there exists a $c \in X$ on none of whose neighbourhoods V $\phi|V$ is one-to-one. This means there are distinct points x_V and y_V in each V and an $s_V \in G$ such that $s_V x_V = y_V$. Then for some $s \in G$ the collection of those V for which $s_V = s$ must be cofinal in the neighbourhood filter of c, that is, any neighbourhood U of c contains some V with $s_V = s$. Call these neighbourhoods W. Since the two (Moore-Smith) sequences x_W and y_W both converge to c, one has $sc = s(\lim x_W) = \lim sx_W = \lim y_W = c$. However, $sc = c$ only holds for the unit of G which contradicts the assumption that all x_V and y_V are distinct. Therefore, any $x \in X$ has a neighbourhood V which is mapped one-to-one. V can be taken as open, and as ϕ is continuous and open $\phi|V$ is a homoemorphism. Finally, $\phi|sV$ is also a homeomorphism for each $s \in G$, $U = \phi V$ is an open neighbourhood of $\phi x \in E$ and $\phi^{-1}U = \bigcup sV$. This completes the proof.

The next item to be considered is the construction of a quasi-covering space of a completely regular E from a quasi-covering space of its βE. In this, the following notion will be employed. If (Y, ψ, W) is a fibre space and E a dense subspace of W, then the fibre space (X, ϕ, E), where $X = \psi^{-1}E$ and $\phi = \psi|X$, will be called the restriction of (Y, ψ, W) to the base E. The group $G(\psi)$, since it transforms X into itself, induces a subgroup of $G(\phi)$ by restriction to X. Obviously, if $G(\psi)$ acts transitively on each fibre, then $G(\phi)$ does too.

LEMMA 4. *If $(Y, \psi, \beta E)$ is a regular quasi-covering space with compact Y, then its restriction (X, ϕ, E) to the base E is again a regular quasi-covering space, with $\beta X = Y$ and $G(\phi)$ induced by $G(\psi)$.*

Proof. Let U be an evenly covered open neighbourhood in βE of $x \in E$ and U' the disjoint open sets into which $\psi^{-1}U$ then splits. For the neighbourhood $V = U \cap E$ of x in E one then has $\phi^{-1}V = \psi^{-1}V = (\psi^{-1}U) \cap X$, and this is the union of the disjoint open $U' \cap X$ in X each of which is mapped homeomorphically onto V by ϕ. Hence (X, ϕ, E) is a quasi-covering space and thus, of course, a regular one.

To obtain $\beta X = Y$, use will be made of completely regular filters. A filter $\mathfrak{A}$ on a space S is called completely regular if for any $A \in \mathfrak{A}$ there exsits some $f \in \mathbf{C}(S)$ such that $0 \leqslant f \leqslant 1$, $f^{-1}\{0\} \in \mathfrak{A}$ and $f = 1$ outside A. If K is a compact space containing S densely, then each $z \in K - S$ determines a filter $\mathfrak{T}(z)$ on S, its trace filter, consisting of the sets $V \cap S$ where V ranges over the neighbourhoods of z in K. These $\mathfrak{T}(z)$ are completely regular filters and $K = \beta S$ holds exactly if they are all maximal completely regular (**4**, ch. 9, §1). In this case, the $\mathfrak{T}(z)$ will be precisely the non-convergent maximal completely regular filters on S, the convergent ones just being the neighbourhood filters of the points of S.

Now, let U be an open neighbourhood of $u \in Y - X$ such that $\psi|U$ is one-to-one. Because $U \cap X$ and $(\psi U) \cap E$ correspond to each other, the trace filter $\mathfrak{T}(u)$ of u on X is mapped by ϕ onto the trace filter $\mathfrak{T}(\psi u)$ of ψu on E. Hence, for any completely regular filter $\mathfrak{R} \supseteq \mathfrak{T}(u)$ on X one has $\phi\mathfrak{R} \supseteq \mathfrak{T}(\psi u)$. However, $\phi\mathfrak{R}$ is again completely regular: for any $A \in \phi\mathfrak{R}$ take a $B \in \mathfrak{R}$ such that its closure in X lies in $U \cap X$ and $\phi B \subseteq A$, and then some $f \in \mathbf{C}(X)$ with $0 \leqslant f \leqslant 1$, $f^{-1}\{0\} \in \mathfrak{R}$ and $f = 1$ outside B. With this f, define g on E by

$$gx = \begin{cases} 1 \text{ if } x \notin (\psi U) \cap E \\ fz \text{ if } x = \phi z,\ x \in (\psi U) \cap E. \end{cases}$$

This g is continuous, has value 1 outside ϕB and hence outside A, and satisfies $0 \leqslant g \leqslant 1$ and $g^{-1}\{0\} \in \phi\mathfrak{R}$. Thus, $\phi\mathfrak{R}$ is completely regular. But now, $\mathfrak{T}(\psi u)$ is maximal completely regular because the extension of E considered is βE; therefore, $\phi\mathfrak{R} = \mathfrak{T}(\psi u) = \phi\mathfrak{T}(u)$ and as $\mathfrak{T}(u)$ contains a set on which ϕ is one-to-one, this gives $\mathfrak{R} = \mathfrak{T}(u)$. Since $\mathfrak{R} \supseteq \mathfrak{T}(u)$ was arbitrary completely regular, $\mathfrak{T}(u)$ is hereby seen to be maximal and this shows $\beta X = Y$.

Finally, any $s \in G(\phi)$ can be uniquely extended to an automorphism s' of $Y = \beta X$ and from $s'u = s'(\lim \mathfrak{T}(u)) = \lim s\mathfrak{T}(u)$ for any $u \in Y - X$ one obtains $\psi(s'u) = \psi u$, that is, $s' \in G(\psi)$. Hence, $G(\phi)$ is the restriction of $G(\psi)$ to X, and this completes the proof.

There is a partial converse to Lemma 4. If (X, ϕ, E) is a fibre space with completely regular X and E, then ϕ has a unique continuous extension ψ: $\beta X \to \beta E$ with $\psi(\beta X) = \beta E$ (**7**). The fibre space $(\beta X, \psi, \beta E)$ will be called the extension of (X, ϕ, E) to the base βE. Each $s \in G(\phi)$ has an extension

to an automorphism of βX which, by the same argument as in the last paragraph, belongs to $G(\psi)$ such that $G(\phi)$ induces a subgroup of $G(\psi)$. A good deal more can be said if further conditions are assumed for (X, ϕ, E).

LEMMA 5. *If (X, ϕ, E) is a finite regular quasi-covering space with connected X such that each non-convergent maximal completely regular filter on E contains an evenly covered open set, then its extension $(\beta X, \psi, \beta E)$ to the base βE is again a regular quasi-covering space and $G(\psi)$ is induced by $G(\phi)$.*

Proof. If G' denotes the continuous extension of $G = G(\phi)$ to βX, then since $G' \subseteq G(\psi)$, ψ induces a continuous mapping of the compact orbit space $\beta X/G'$ onto βE such that E, which is also a subspace of $\beta X/G'$ (**4**, ch. I, §9) remains pointwise fixed. Therefore $\beta X/G' = \beta E$ by the maximality of βE and thus G' acts transitively on each fibre of $(\beta X, \psi, \beta E)$. Next, no $s' \in G'$ other than the unit has any fixed point. Such a point would have to be a $u \in \beta X - X$ and if $s'u = u$, $s' \in G'$ and not the unit, it would have arbitrarily small neighbourhoods U with $s'U = U$, since s' is of finite order, and hence its trace filter $\mathfrak{T}(u)$ on X would have a basis of sets V with $sV = V$. However, by continuity one has $\mathfrak{T}(\psi u) \subseteq \phi\mathfrak{T}(u)$, and by hypothesis $\mathfrak{T}(\psi u)$ contains an evenly covered open W. Then, $\phi^{-1}W \in \mathfrak{T}(u)$ where $\phi^{-1}W = \bigcup W_k$ with finitely many disjoint open W_k such that $\phi|W_k$ is a homeomorphism; it follows that $W_k \in \mathfrak{T}(u)$ for some k (**2**), but of course there is no $V \subseteq W_k$ with $sV = V$. This contradiction proves $s'u \neq u$.

One now obtains, by the same argument as in the proof of Lemma 3, that $(\beta X, \psi, \beta E)$ is a quasi-covering space; since $G' \subseteq G(\psi)$ is transitive on each fibre this implies regularity, and since X is connected one has $G' = G(\psi)$.

Remark. If (X, ϕ, E) satisfies the hypothesis in Lemma 5, then, by this lemma, E is the union of finitely many evenly covered sets since βE is compact. Conversely, this condition implies the hypothesis in Lemma 5, at least if E is a normal space since for such E any finite open covering of E is induced by one of βE. In particular, therefore, if E is normal and the union of finitely many simply connected open sets (in the sense of Chevalley) then, for any finite regular covering space (X, ϕ, E) the extension $(\beta X, \psi, \beta E)$ is a regular quasi-covering space, with $G(\psi)$ induced by $G(\phi)$. More generally, the same holds if E, not necessarily normal, has any compact extension K such that each $u \in K - E$ has an open neighbourhood for which $V = U \cap E$ is simply connected, for then any maximal completely regular filter on E will converge to some such u and hence contain the corresponding V which in turn will be evenly covered by any quasi-covering space (X, ϕ, E). Similarly, the above statement concerning $(\beta X, \psi, \beta E)$ is true if E is a finite dimensional separable metric space, for according to (**6**) X then contains a finite number of open W_i such that $\phi|W_i$ is a homeomorphism and $E = \bigcup \phi W_i$, and thus E is normal and the union of finitely many evenly covered open sets, the latter because $\phi^{-1}\phi W_i = \bigcup sW_i$, $s \in G(\phi)$.

4. Extensions of function rings. Let $\mathbf{C}$ denote any directly indecomposable function ring and G an arbitrary group.

PROPOSITION 1. *There exists a normal extension* $\mathbf{E}$ *of* $\mathbf{C}$ *which has* G *as a group of automorphisms over* $\mathbf{C}$.

Proof. If E is the maximal ideal space of $\mathbf{C}$ and (X, ϕ, E) the fibre space of Lemma 1, then, since E and X are both completely regular, Lemma 2 shows that $\mathbf{E} = \mathbf{C}(X)$ has the required properties if one identifies $\mathbf{C}$ with $\mathbf{C}(E)^*$.

An immediate consequence of Proposition 1 is that any directly indecomposable function ring $\mathbf{C}$ possesses directly indecomposable extensions $\mathbf{E}$ such that each element of $\mathbf{E}$ is algebraic over $\mathbf{C}$, for to obtain such extensions one only has to take a normal extension of C with some finite group. In other words, there is no such thing as algebraic closure in the class of all directly indecomposable function rings.

Another observation that can be made here is that if a directly indecomposable function ring $\mathbf{C}$ contains a maximal ideal which is not the sum of two closed ideals whose intersection is the zero ideal, then for any natural number n there exists a normal extension $\mathbf{E}$ of $\mathbf{C}$ such that the invariance group of $\mathbf{C}$ in $\mathbf{E}$ is isomorphic to the symmetric group S_n of n objects. First, the condition for the maximal ideals stated means that the maximal ideal space E of $\mathbf{C}$ contains a point a such that $E - \{a\}$ is connected. Then the remark following the proof of Lemma 1 shows that for (X, ϕ, E), constructed with any group of order n, one has $G(\phi) \cong S_n$ and since X is compact here, (1) in §1 implies that $\mathbf{C}(E)^* \cong \mathbf{C}$ has $\bar{G}(\phi) \cong S_n$ as its invariance group in $\mathbf{E} = \mathbf{C}(X)$.

The extensions obtained from Lemma 1 are normal but not Galois. Examples for the latter arise from regular quasi-covering spaces (X, ϕ, E) with connected X. There, $G(\phi)$ is the only group $H \subseteq G(\phi)$ acting transitively on each fibre and if X (and thus E) is compact or, for instance, completely regular and satisfying the first axiom of countability, then by Lemma 2, by (1) or (2) in §1 and by (5) $\bar{G}(\phi)$ is the invariance group of $\mathbf{C}(E)^*$ in $\mathbf{C}(X)$ and no proper subgroup of $\bar{G}(\phi)$ has $\mathbf{C}(E)^*$ as its ring of invariants. In other words, $\mathbf{C}(X)$ is then a Galois extension of $\mathbf{C}(E)^*$ with group $\bar{G}(\phi)$ which is, furthermore, isomorphic to $G(\phi)$.

If a Galois extension $\mathbf{E}$ of $\mathbf{C} = \mathbf{C}(E)$ is given, in the manner just described, by some regular quasi-covering space (X, ϕ, E) it will be called the Galois extension of $\mathbf{C}$ associated with (X, ϕ, E). The finite Galois extensions of this type have a certain property which can best be formulated by means of the following concept, borrowed from the ideal theory of algebraic number fields: a finite Galois extension $\mathbf{E}$ of $\mathbf{C}$ is said to decompose fully the maximal ideal $\mathfrak{m}$ in $\mathbf{C}$ if there are exactly as many maximal ideals $\mathfrak{M} \supseteq \mathfrak{m}$ in $\mathbf{E}$ as the group of $\mathbf{E}$ over $\mathbf{C}$ has elements. It is clear that all Galois extensions of $\mathbf{C}(E)$ associated with finite regular quasi-covering spaces (X, ϕ, E) fully decompose each fixed maximal ideal of $\mathbf{C}(E)$. However, not all finite Galois extensions of $\mathbf{C}(E)$

with this latter property are associated with regular quasi-covering spaces of E. Take, for instance, E to be an open annular region in the plane, (X, ψ, E) its regular covering space with group of order 2, and $\mathfrak{T}_1$ and $\mathfrak{T}_2$ two maximal completely regular filters on Y above the same maximal completely regular filter on E. Now, let $\mathbf{E}$ be the ring of all $f \in \mathbf{C}(Y)$ with $\lim f\mathfrak{T}_1 = \lim f\mathfrak{T}_2$. $\mathbf{E}$ is then a Galois extension of $\mathbf{C} = \mathbf{C}(E)^*$, its group also being of order 2. However, it is not associated with any regular quasi-covering space of E since this would have to have group of order 2, and (Y, ψ, E) is the only such quasi-covering space whereas $\mathbf{E} \neq \mathbf{C}(Y)$.

This observation suggests that in order to describe at least a class of finite Galois extensions of $\mathbf{C}(E)$ associated with regular quasi-covering spaces of E by means of simple algebraic conditions one has to assume full decomposition for all maximal ideals. With this, the following characterization can be obtained.

Proposition 2. *Let E be a connected completely regular space. Then, a Galois extension* $\mathbf{E}$ *of* $\mathbf{C} = \mathbf{C}(E)$ *has finite group and fully decomposes each maximal ideal in* $\mathbf{C}$ *if and only if it is associated with a finite regular quasi-covering space (X, ϕ, E), with connected X, in which each non-convergent maximal completely regular filter on E contains an evenly covered open set.*

Proof. In the case of compact E, in which there are no non-convergent maximal completely regular filters, this is obvious from **(3; 1)** and Lemma 3. If E is not compact, one can first use the proposition for βE since $\mathbf{C}(E) = \mathbf{C}(\beta E)$. Hence, any extension $\mathbf{E}$ of the kind stated is associated with a regular quasi-covering space $(Y, \psi, \beta E)$ to which Lemma 4 can be applied; this gives a regular quasi-covering space (X, ϕ, E) with which $\mathbf{E}$ is also associated, and because of its origin and $Y = \beta X$, (X, ϕ, E) has the additional propreties concerning the maximal completely regular filters. Also, X is connected since $Y = \beta X$ is. Conversely, if (X, ϕ, E) satisfies all the conditions listed, then Lemma 5 shows that each maximal ideal in $\mathbf{C}(E)^*$ is contained in precisely as many maximal ideals of $\mathbf{C}(X)$ as $G(\phi)$ has elements. This means the extension of $\mathbf{C} = \mathbf{C}(E)$ associated with (X, ϕ, E) fully decomposes each maximal ideal of $\mathbf{C}$.

The condition for (X, ϕ, E) concerning the maximal completely regular filters on E prevents Proposition 2 from giving a complete description of the Galois extensions of $\mathbf{C}(E)$ which are associated with finite regular quasi-covering spaces of E. However, the remark following Lemma 5 shows that for certain types of spaces this condition is redundant. Hence one has:

Corollary 1. *If E possesses a compact extension K such that each $u \in K - E$ has an open neighbourhood U for which $U \cap E$ is simply connected or if E is a finite dimensional separable metric space, then the extensions of* $\mathbf{C} = \mathbf{C}(E)$ *associated with finite regular quasi-covering spaces of E are precisely the finite Galois extensions of* $\mathbf{C}$ *which fully decompose each maximal ideal of* $\mathbf{C}$.

In Proposition 2, attention is paid to the way in which the function ring **C** is concretely represented on some space. Independent of such representation of **C** one has:

COROLLARY 2. *If the finite Galois extension* **E** *of* **C** *fully decomposes each maximal ideal of* **C**, *then its group G permutes the maximal ideals of* **E** *above each maximal ideal of* **C** *transitively and* **E** *is finitely generated over* **C**.

Proof. **E** is associated with a regular quasi-covering space (X, ϕ, E) where X and E are the maximal ideal spaces of **E** and **C**. The first part follows immediately from this. For the second part, one observes first that E is the union of finitely many evenly covered V_t and the parts V_{tk} into which the $\phi^{-1}V_t$ split form an open covering of X. Then, let $f_{tk} \in \mathbf{C}(X)$ be a decomposition of the unity, subordinate to this open covering, that is, $0 \leq f_{tk} \leq 1$, $f_{tk} = 0$ outside V_{tk} and $\sum f_{tk} = 1$ (**4**, ch. IX, §4) and take h_{tk} such that $h_{tk}^2 = f_{tk}$. Now, for any $g \in \mathbf{C}(X)$ put $g_{tk} = \sum \bar{s}(gh_{tk})$, $s \in G(\phi)$; one then has $g_{tk}h_{tk} = gh_{tk}^2$ and hence $g = \sum g_{tk}h_{tk}$ where $g_{tk} \in \mathbf{C}(E)^*$. Thus, the h_{tk} generate $\mathbf{C}(X)$ over $\mathbf{C}(E)^*$.

5. Concluding remarks. In view of Corollary 1 of Proposition 2 one might ask whether the restriction on (X, ϕ, E) in this proposition is not, in fact, always redundant, in which case one would have a simple algebraic characterization of all Galois extensions of a function ring $\mathbf{C}(E)$ which are associated with finite quasi-covering spaces of E. This question is equivalent with the following problem: if G is a finite group of automorphisms of a connected completely regular X such that no $s \in G$ other than the unit has any fixed points, does the continuous extension of G to βX have the same property? In order to show that the answer is positive it would be sufficient to prove, for any such X and G, the existence of some compact extension of E onto which G can be continuously extended without losing its particular property; however, whilst this can be done for various special types of spaces, we do not know whether it is possible in general.

Quite apart from this problem, a certain "external" characterization of the Galois extensions of $\mathbf{C} = \mathbf{C}(E)$ which are associated with finite regular quasi-covering spaces of E can be given in the following way: the methods in §3 readily show that any Galois extension **E** of **C** which has finite group G and decomposes fully each fixed maximal ideal of **C** can be represented as the ring of (some) functions on a regular quasi-covering space (X, ϕ, E) such that G is induced isomorphically by $G(\phi)$. Therefore, a finite Galois extension **E** of **C** is associated with a regular quasi-covering space if and only if it fully decomposes each fixed maximal ideal of **C** and is not contained in any larger Galois extension of **C** with the same property whose group isomorphically induces the group of **E**.

Another question arising from Proposition 2 is whether a similar treatment be possible in the case of infinite groups. For this the concept of full decom-

position of a maximal ideal has to be extended first, and it is natural to do this by calling a maximal ideal $\mathfrak{m}$ in $\mathbf{C}$ fully decomposed by the Galois extension $\mathbf{E}$ of $\mathbf{C}$ with the group G if for any maximal ideal $\mathfrak{M} \supseteq \mathfrak{m}$ in E $s\mathfrak{M}$, $s \in G$, ranges over all maximal ideals above in such that $s\mathfrak{M} \neq t\mathfrak{M}$ if $s \neq t$. However, a Galois extension $\mathbf{E}$ of $\mathbf{C} = \mathbf{C}(E)$ associated with a regular quasi-covering space (X, ϕ, E) whose $G(\phi)$ is infinite cannot even fully decompose any fixed maximal ideal of $\mathbf{C}$, for each fibre $\phi^{-1}x$, $x \in E$, of the extension $(\beta X, \psi, \beta E)$ must contain points from $\beta X - X$ since otherwise $\phi^{-1}x = \psi^{-1}x$, which would mean $\phi^{-1}x$ is closed in βX besides being discrete and thus finite. Therefore, the concept of full decomposition of maximal ideals is useless for the description of the infinite Galois extensions of a function ring $\mathbf{C}(E)$ which are associated with regular quasi-covering spaces of E, whereas, on the other hand, such extensions do exist for suitable E.

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LATTICE THEORY OF GENERALIZED PARTITIONS

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1. Introduction. In **(1)** the lattice of all equivalence relations on a set S was studied and many important properties were established. In **(2)** and **(3)** the lattice of all geometries on a set S was studied and it was shown to be a universal[1] lattice which shares many properties with the lattice of equivalence relations on S. In this paper we shall give the definition of a partition of type n and investigate the lattice formed by all partitions of type n on a fixed set S. It will be seen that a partition of type one on S can be considered as an equivalence relation on S and similarly a partition of type two on S can be considered as a geometry on S as defined in **(2)**. Thus we shall obtain a unified theory of lattices of equivalence relations, lattices of geometries and partition lattices of higher types. We shall observe that most of the properties which hold for the partition lattices of type one and two hold for partition lattices of any type. We shall first show that a partition lattice of type n on a set S is a complete point lattice which is isomorphic to the lattice of subspaces of a suitably chosen geometry. A characterization of the lattice of equivalence relations on S was given in **(1)**. We shall give a similar characterization of the lattice of all geometries on S (that is, the lattice of partitions of type two on S) by characterizing the geometries whose lattices of subspaces are isomorphic to the lattice of geometries. We shall then show that the lattices of partitions of any type are complemented and special properties of these complements will be investigated. It shall further be shown that these lattices are simple and the groups of automorphisms will be constructed. Finally we shall investigate the complete homomorphisms of lattices of subspaces of geometries and characterize them in terms of polygons in the geometry. We shall conclude by stating some unsolved problems.

2. Generalized partitions.

Definition 1. A partition of type n, $n > 1$, on the set S consisting of n or more elements is a collection of subsets of S such that any n distinct elements of S are contained in exactly one subset and every subset contains at least n distinct elements.

It can be seen that the subsets of a partition of type one on S define an equivalence relation on S and vice versa. Similarly a geometry on S is equivalent to a partition of type two on S if we consider the lines defining the geometry as the subsets which form the partition.

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[1] Any finite lattice is isomorphic to a sublattice of the lattice of all geometries on some finite set.

We recall that a subset T of S is said to be a subspace of a geometry G on S if with any two distinct elements of T the line containing these elements is in T. Thus the set of all subspaces of a geometry G on S ordered under set inclusion forms a complete lattice.

We shall refer to the subsets defining a partition as blocks. A block of a partition of type n is said to be non-trivial if it consists of at least $n+1$ distinct elements. Otherwise we shall call the block trivial. We shall represent a partition P by the set of its non-trivial blocks, $P = \{S_\alpha\}$.

Let us now order the set of all partitions of type n on S by defining $P_1 \leqslant P_2$ if and only if every block of P_1 is contained in a block of P_2. Under this ordering it is a partially ordered set which is closed under arbitrary intersections. To see this we just have to note that if $\{P_\alpha \mid \alpha \in A\}$ is any set of partitions of type n on S then the partition, whose blocks containing any n prescribed points $x_1, x_2, \ldots, x_n$ of S are obtained by intersecting the corresponding blocks of $P_\alpha, \alpha \in A$, is the g.l.b. of $\{P_\alpha \mid \alpha \in A\}$. Since this partially ordered set of partitions has a unit element we conclude that it is a complete lattice. We shall denote it by $L\,P_n(S)$. To simplify the statements of the theorems we shall first agree to define $L\,P_n(S)$ to consist of a single element if S contains less than n elements, secondly, let $L\,P_0(S)$ denote the Boolean algebra of all subsets of S. It can be seen that $L\,P_n(S)$, $|S| > n \geqslant 1$, is a point lattice and that the points are partitions consisting of only one non-trivial block and this block contains $n+1$ distinct elements.

THEOREM 1. *$L\,P_n(S)$ is isomorphic to the lattice of subspaces of some geometry.*

Proof. The result holds for $n = 0$, since $L\,P_0(S)$ is isomorphic to the lattice of subspaces of the geometry on S in which every line consists of exactly two points. By our previous convention the result holds trivially for $L\,P_n(S)$, $n \geqslant 1$, if S consists of n or less elements. If S consists of more than n elements then the union in $L\,P_n(S)$ of any two distinct points $P_1 = \{(a_1, a_2, \ldots, a_{n+1})\}$ and $P_2 = \{(b_1, b_2, \ldots, b_{n+1})\}$ is either a partition with only one non-trivial block and then this block consists of $n+2$ elements, or it is a partition consisting of two non-trivial blocks, that is, $P_1 \cup P_2 = \{(a_1, a_2, \ldots, a_{n+1}), (b_1, b_2, \ldots, b_{n+1})\}$. In either case $P_1 \cup P_2$ covers P_1 and P_2. This implies that the sets of points of $L\,P_n(S)$, which are contained in unions of two distinct points, form the set of lines for a geometry on the set of points of $L\,P_n(S)$. We now observe that if T is a subspace of this geometry and if for a point Q of $L\,P_n(S)$ we have that $Q \leqslant \cup \{P | P \in T\}$, then $Q \in T$. This implies that $L\,P_n(S)$ is isomorphic to the lattice of subspaces of the geometry defined by the unions of pairs of points of $L\,P_n(S)$.

3. Characterization of LG(S). We shall now characterize the geometries whose lattices of subspaces are isomorphic to the partition lattices of type two on S.

We shall introduce some concepts which are essential for the following theory.

Definition 2. Two distinct points p and q of a geometry G are said to be *related* if the line defined by p and q is non-trivial.

Definition 3. The points p and q of a geometry G are said to be *close* if p is equal to q, p is related to q, or there exists a point t of G such that p is related to t and t is related to q.

Definition 4. Let the line l of G consist of the four distinct points p_1, p_2, p_3, p_4, and let π denote the set consisting of the three collinear points p_1, p_2, and p_3. Then a point q of G is said to be *close* to π if one of the two following conditions holds:

(i) q is equal to p_1, p_2, or p_3,

(ii) q is close to p_1, p_2, p_3; q is distinct from p_4 and not related to p_4.

For further discussion π will denote a triplet of distinct collinear points of G.

Definition 5. π_1 is said to be *close* to π_2 if every point in π_1 is close to π_2.

THEOREM 2. *Let L be the lattice of subspaces of a geometry G on W and let W consist of four or more points. Then L is isomorphic to the lattice of all geometries on some set S if and only if G satisfies the following five axioms*:

Axiom 1. The non-trivial lines of G consist of four points and every point is contained in at least one non-trivial line.

Axiom 2. If a point p is close to π_1 and π_1 is close to π_2 then p is close to π_2.

Axiom 3. If π_1 is close to π_2 then π_2 is close to π_1.

Axiom 4. If π_1, π_2, π_3 are distinct then there exists a point p such that p is close to π_1, π_2, π_3.

Axiom 5. Let l be a non-trivial line and let p be a point which is not on this line but is close to every point on this line, then p is related to exactly two points of l.

To show that L is isomorphic to the lattice of all geometries on some set S we have to show that there exists a one-to-one mapping of the set W onto the set of points of $LG(S)$ ($LG(S) = L\,P_2(S)$) and that this mapping preserves lines. To do this we shall introduce the concept of a *star* of G. Let π be a triplet of distinct collinear points of G. Then the set of all points which are close to π will be called the star of G defined by π. This set will be denoted by $\Delta(\pi)$. We shall show that the lattice of all geometries on the set of stars of G is isomorphic to L. We know that a point of $LG(S)$ is a geometry with only one non-trivial line and this line consists of three points. Thus every point of $LG(S)$ is characterized by the three elements of S which are contained in its non-trivial line. Therefore we first have to establish a one-to-one mapping of the set W onto the set of all triplets consisting of distinct stars of G. We shall do this by showing that every point of G is contained in exactly three distinct stars and that any three distinct stars have exactly one point of G in common. The proof consists of the following lemmas.

Lemma 1. *If the point p is close to π_1 then there exists π_2 such that p is contained in π_2 and π_2 is close to π_1.*

Proof. Let π_1 consist of the distinct points p_1, p_2, p_3 and let the fourth collinear point of this line l be p_4. If p is contained in l then p must be equal to p_1, p_2, or p_3 since by Definition 4 the fourth collinear point p_4 is not close to π_1. Thus we may set $\pi_2 = \pi_1$ since then p is contained in π_2 and by Definition 4 and Definition 5 we see that π_1 is close to π_1. Let us now assume that p is not on the line l but is related to a point of π_1. Let this point be p_1 as indicated in Figure 1. Then p_4 is related to p_1 and p_1 is related to every point of the

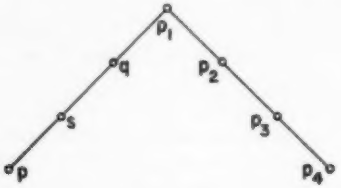

Figure 1

line defined by p and p_1. Thus by Definition 3, p_4 is close to every point on this line and therefore by Axiom 5, p_4 is related to exactly two of the four distinct points of this line. We know that p_4 is related to p_1 and let us denote the second point to which it is related by s. Since the line defined by p and p_1 consists of four points there must exist a point q on this line which is distinct from p and is not related to p_4. Let π_2 consist of p, p_1, and q. We see that p is contained in π_2 and we shall show that π_2 is close to π_1. By Definition 5 we have to show that every point of π_2 is close to π_1. p_1 is contained in π_1 and therefore by Definition 4 close to π_1. p and q are close to every point in π_1 and not related to p_4. Thus by Definition 4 they are close to π_1. It follows that π_2 is close to π_1. We may now assume that p is not related to any point on the line l. Since p is close to p_1 there exists a point t of G such that p is related to t and t is related to p_1 as shown in Figure 2. Using the result of the previous case we know that there exists a triplet π_3 on the line defined by p_1 and t such that t is contained in π_3 and π_3 is close to π_1. Let us denote

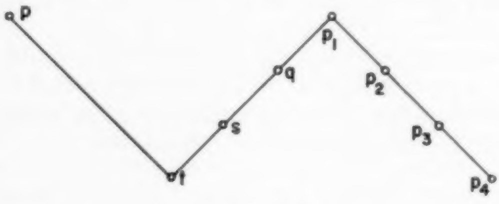

Figure 2

the point of this line which is not contained in π_3 by s as we did in the previous case. We recall that s is related to p_4. Thus p cannot be related to s since otherwise p is related to s and s related to p_4 which implies that p is close to p_4 and therefore close to every point on l. From this we would conclude that p is related to exactly two points on the line l, contrary to assumption. But then p is close to π_3 and is related to t which is contained in π_3. Using again the result of the previous case there exists a triplet π_2 on the line defined by p and t such that p is contained in π_2 and π_2 is close to π_3. Now we have that p is contained in π_2, π_2 is close to π_3, and π_3 is close to π_1. Thus by Axiom 2 we conclude that π_2 is close to π_1. This completes the proof of Lemma 1.

LEMMA 2. *Any there distinct collinear points contained in a star define the star.*

Proof. Let π_1 be contained in $\Delta(\pi_2)$. Then every point of π_1 is close to π_2 and therefore π_1 is close to π_2. By Axion 3, π_2 is close to π_1 and therefore by Axiom 2 every point which is close to π_1 is close to π_2 and vice versa. From this it follows that $\Delta(\pi_1) = \Delta(\pi_2)$.

LEMMA 3. *Every point p of G is contained in at least three distinct stars.*

Proof. By Axiom 1 a point p of G is contained in at least one non-trivial line l and this line consists of four points. There are exactly three distinct triplets π_1, π_2, and π_3 of l which contain p. The fourth collinear point of l which is not contained in π_1 is by Definition 4 not contained in $\Delta(\pi_1)$. But this point is contained in π_2 and π_3 and therefore it is contained in $\Delta(\pi_2)$ and $\Delta(\pi_3)$. Thus $\Delta(\pi_1)$ is distinct from $\Delta(\pi_2)$ and $\Delta(\pi_3)$. Similarly we show that $\Delta(\pi_2)$ is distinct from $\Delta(\pi_3)$. This shows that there are at least three distinct stars which contain p.

LEMMA 4. *There are exactly three distinct stars contining every point p of G.*

Proof. Let p be contained in a non-trivial line l and let π_1, π_2, and π_3 be the distinct triplets of l which contain p. By the previous result we know that the stars $\Delta(\pi_1)$, $\Delta(\pi_2)$, and $\Delta(\pi_3)$ are distinct. Let p be contained in some star $\Delta(\pi)$; we shall show that $\Delta(\pi)$ is equal to $\Delta(\pi_1)$, $\Delta(\pi_2)$, or $\Delta(\pi_3)$. Note that if p is contained in $\Delta(\pi)$ then by Lemma 1 there exists a triplet π' such that p is contained in π' and π' is close to π. If π' is contained in the line l then it must be equal to π_1, π_2, or π_3 and therefore by Axiom 3 and Axiom 2 we conclude that $\Delta(\pi)$ is equal to $\Delta(\pi_1)$, $\Delta(\pi_2)$, or $\Delta(\pi_3)$. Thus we may assume that π' is contained in a non-trivial line l' and that l' is distinct from l. Let us denote the point of l' which is not contained in π' by q. Since p is contained in π' and therefore in l' we see that q is related to p and p is related to every element of the line l. Thus q is close to every point on the line l and therefore related to exactly two points of l. We know that q is related to p. Let the second point to which p is related be denoted by s. One of the triplets π_1, π_2, or π_3 does not contain the point s, say π_1. Then π_1 is close to π' since p

is contained in π_1 and the two remaining points of π_1 are close to every point in π' and not related or equal to q. From the fact that π_1 is close to π' it follows by Axiom 3 and Axiom 2 that $\Delta(\pi_1) = \Delta(\pi')$; this proves Lemma 4.

By Axiom 4 any three distinct stars have at least one point in common. The next lemma will show that there cannot be more than one such point.

LEMMA 5. *Any three distinct stars have exactly one point in common.*

Proof. Let p and q be distinct points of L. We shall show that the three distinct stars which contain p cannot all contain q. Let p be contained in the non-trivial line l and let π_1, π_2, and π_3 be the distinct triplets of l which contain p. These triplets define the three distinct stars which contain p. If q is also contained in the line l then q is not contained in one of these triplets. Let this triplet be π_1. Then q is not contained in the star $\Delta(\pi_1)$ since q is the point of l which is not contained in π_1. Thus we may assume that p and q are not related, which implies that q cannot be close to π_1, π_2, and π_3. Since if q would be close to π_1, π_2, and π_3 then q would be close to every point on l and therefore q would be related to exactly two elements of l. But then q would be related to an element of l which is not contained in one of the triplets π_1, π_2, or π_3 and therefore q would not be close to one of these triplets, contrary to the assumption. Thus q is not contained in one of the three stars which contain p. This proves Lemma 5.

So far we have shown that there exists a one-to-one mapping of the set W onto the set of points of $LG(S)$, where S is the set of stars of G. Let us denote this mapping by θ.

LEMMA 6. *The mapping θ preserves lines.*

Proof. Let l be a non-trivial line of G. Then l contains four distinct triplets $\pi_1, \pi_2, \pi_3, \pi_4$ and these triplets define the four distinct stars $\Delta_1, \Delta_2, \Delta_3, \Delta_4$, respectively. Every point of the line l is contained in three of these triplets and therefore in three of these stars. Under the mapping θ the line l is mapped into the line l' of $LG(S)$ which consists of the four points $\{(\Delta_1, \Delta_2, \Delta_3)\}$, $\{(\Delta_1, \Delta_2, \Delta_4)\}$, $\{(\Delta_1, \Delta_3, \Delta_4)\}$, and $\{(\Delta_2, \Delta_3, \Delta_4)\}$. This shows that every point on the line l is mapped into a point of the corresponding line l' of $LG(S)$. Conversely, let l' be a non-trivial line of $LG(S)$ and let this line consist of the four points $\{(\Delta_1, \Delta_2, \Delta_3)\}$, $\{(\Delta_1, \Delta_2, \Delta_4)\}$, $\{(\Delta_1, \Delta_3, \Delta_4)\}$ and $\{(\Delta_2, \Delta_3, \Delta_4)\}$. Let $\{(\Delta_1, \Delta_2, \Delta_3)\}$ and $\{(\Delta_1, \Delta_2, \Delta_4)\}$ be mapped onto the points p and q respectively. Let p be contained in a non-trivial line l of G. We know that the triplets π_1, π_2, π_3 of l which contain p define the stars $\Delta_1, \Delta_2, \Delta_3$, respectively. Then q is contained in Δ_1 and Δ_2 and therefore q is close to every point on l. Thus q is related to exactly two points of l. Thus q has to be related to p since otherwise one of the triplets π_1 or π_2 would not contain a point of l to which q is related and therefore q would not be close to π_1 or π_2, contrary to assumption. Thus p and q are related. Without loss of generality we may assume that p and q are contained in l and that $\pi_1, \pi_2, \pi_3, \pi_4$ are the distinct

triplets of l. Then π_1, π_2, π_3, and π_4 define the stars Δ_1, Δ_2, Δ_3, and Δ_4, respectively. Clearly every three distinct triplets of the set $\pi_1, \pi_2, \pi_3, \pi_4$ have a point in common and this point is contained in l. This shows that every point of the line l' is mapped into a point on the corresponding line l. Thus the mapping preserves lines and we conclude that L is isomorphic to $LG(S)$.

A straightforward computation shows that the five axioms of Theorem 2 hold in $LG(S)$ if S consists of four or more elements.

Thus we have completed the proof.

4. Lattice theoretic properties of $L\,P_n(S)$. We shall now study the lattice theoretic properties of $L\,P_n(S)$.

THEOREM 3. *For each given point P of $L\,P_n(S)$ and a given integer m, $n > m > 0$, there exists a complete sublattice L of $L\,P_n(S)$ such that*

(i) *L is isomorphic to $L\,P_m(S - \{a_1 \vee a_2 \vee \ldots \vee a_{n-m}\})$,*

(ii) *L contains the point P,*

(iii) *the unit and zero elements of L and $L\,P_n(S)$ coincide.*

Proof. Let a point $P = \{a_1 \vee a_2 \vee \ldots \vee a_{n+1}\}$ of $L\,P_n(S)$ be given. If $m = n$ then the theorem is trivially satisfied. Otherwise we let M denote the set $\{a_1, a_2, \ldots, a_{n+1}\}$. Let us now consider all the elements of $L\,P_n(S)$ whose non-trivial blocks contain M and let us denote this set by L. It can be seen that P, O, and I are in L and that L is closed under arbitrary intersections in $L\,P_n(S)$. Furthermore for any subset $\{A_\alpha\}$ of L and C in $L\,P_n(S)$ such that $A_\alpha < C$ there exists C' in L such that $A_\alpha < C' < C$. To obtain C' from C we just remove all the non-trivial blocks of C which do not contain the set M and replace them by the necessary trivial blocks. From this we conclude that $\cap\{A \mid A > A_\alpha\} = \cup\{A_\alpha\}$ is an element of L, which shows that L is a complete sublattice of $L\,P_n(S)$.

To show that L is isomorphic to a partition lattice of type n-m on S-M we observe that after the removal of the set M any two blocks of an element in L can have almost $n - m - 1$ points in common. Thus any element of L can be considered as a partition of type $n - m$ on $S - M$ and to every partition of this type there corresponds a unique element of $L\,P_n(S)$. Since this one-to-one correspondence is order-preserving we conclude that L is isomorphic to $L\,P_{n-m}(S - M)$.

THEOREM 4. *$L\,P_n(S)$ is complemented.*

Proof. $L\,P_0(S)$ is known to be complemented. We shall give a general proof for $n > 1$ which will include the cases previously proven for $n = 1$ and 2. To construct a complement for a partition P of $L\,P_n(S)$, $P \neq I, O$, we let A be a subset of S such that A has at most n points in common with any block of P. The collection of all such sets forms a partially ordered set under set inclusion and by the Maximal Principle it follows that there exists

a maximal element in this partially ordered set. Let us denote this maximal set by M and let us consider the partition $\{M\}$ whose only non-trivial block is M. We first observe that $\{M\} \cap P = 0$. We shall now show that $\{M\} \cup P = I$. Let the block of $\{M\} \cup P$ which contains M be denoted by K. Then $\{M\} \cup P$ consists of K and the blocks of P which are not contained in K. But this will be shown to imply that $K = S$. If there should exist an x in S and not in M then, recalling that M was a maximal set, there must exist $n + 1$ distinct elements $x, x_1, x_2, \ldots, x_n$ which are in $M \vee \{x\}$ and some block B of P. But then $x_1, x_2, \ldots, x_n$ are in K and B and therefore $B \subseteq K$. Thus x is in K and we conclude that $K = S$. Which proves that $L\,P_n(S)$ is complemented.

We observe that the complement of P which was constructed in the previous proof has only one non-trivial block and that we can construct this block so that it contains any n prescribed points of S. Thus if P is not the zero or the unit element of $L\,P_n(S)$ it contains a block which contains at least $n + 1$ distinct elements. We can pick two distinct sets of n elements from this block and for each set construct a complement of P whose non-trivial block contains this set. It can be seen that these complements are distinct. Thus we have proven the following result.

Corollary 1. *If P in $L\,P_n(S)$, $n \geqslant 1$, has a unique complement then $P = I$ or* 0.

Corollary 2. *$L\,P_n(S)$ contains a sublattice isomorphic to a Boolean algebra and every element of $L\,P_n(S)$ has a complement in this sublattice.*

Proof. The corollary holds for $n = 0$. Otherwise we know from Theorem 3 that the set of elements of $L\,P_n(S)$ whose non-trivial blocks contain the fixed set consisting of n distinct elements $a_1, a_2, \ldots, a_n$ forms a sublattice L which is isomorphic to a Boolean algebra. On the other hand, we know from the preceding remarks that every element of $L\,P_n(S)$ has a complement in this sublattice. This completes the proof.

Let us now investigate the homomorphisms of $L\,P_n(S)$, $n \geqslant 1$.

Theorem 5. *There are only trivial homomorphisms of $L\,P_n(S)$, $n \geqslant 1$.*

Proof. In **(1)** Ore showed that $L\,P_1(S)$ has only trivial homomorphisms. This will be shown to imply that $L\,P_n(S)$, $n \geqslant 1$, has only trivial homomorphisms. To see this we recall that if θ is a homomorphism on a point lattice which identifies at least two distinct elements then at least one point has to be mapped into the zero element. Thus a point P of $L\,P_n(S)$ has to be mapped into the zero element. On the other hand, by Theorem 3 we know that there exists a sublattice L of $L\,P_n(S)$ which is isomorphic to $L\,P_1(S)$ and which contains P. But then θ identifies two distinct elements of $L \cong L\,P_1(S - M)$ which implies that all elements of L are identified. Thus, since the zero and unit elements of L and $L\,P_n(S)$ coincide, we conclude that all elements of $L\,P_n(S)$ are identified.

THEOREM 6. *The group of automorphisms of $L\,P_n(S)$ is isomorphic to the symmetric group on S if S consists of more than n elements.*

Proof. The result has been shown to hold for $n = 0, 1, 2$. Our proof will hold for $n > 1$. We note that any automorphism of $L\,P_n(S)$ has to map an element of the form $\{S - p\}$, p in S, onto some other element of the same type. This induces a permutation on the set S and clearly to every permutation on the set S there corresponds an automorphism of $L\,P_n(S)$. Furthermore we know that every element of $L\,P_n(S)$ can be written as a union of intersections of elements of the form $\{S - p\}$. Thus we conclude that the group of automorphisms of $L\,P_n(S)$ is isomorphic to the symmetric group on S.

In previous research and in this paper, it was seen that the concept of a geometry and the lattice of subspaces of this geometry does appear in many mathematical investigations. We shall now show the connection between the non-trivial polygons in the geometry G and the complete homomorphisms of the lattice of subspaces of G.

We shall call a line l of G non-trivial if l consists of more than two points. Let us call a set of non-trivial lines $l_1, l_2, \ldots, l_n$ a non-trivial polygon if, considering $\{l_1\}, \{l_2\}, \ldots, \{l_n\}$ as partitions of type one on S, we have $\{l_1\} \cup \{l_2\} \cup \ldots \cup \{l_n\} = \{l_1 \vee l_2 \vee \ldots \vee l_n\}$. It can be seen that the non-trivial polygons induce an equivalence relation on the points of the geometry G if we define a and b to be in the same equivalence class if and only if a and b can be connected by a non-trivial polygon.

THEOREM 7. *There is a one-to-one order preserving correspondence between the complete homomorphisms of the lattice of G and the subsets of the equivalence classes defined in G by the non-trivial polygons.*

Proof. Let S consist of three or more elements. Then the lattice of subspaces of the geometry G on S, which has only one line $l = S$ has only trivial homomorphisms. Note that if $S = \{p_1, p_2, \ldots, p_n\}$ then the lattices of subspaces is $\{S > p_1, p_2, \ldots, p_n > \phi\}$, but this lattice is known to have only trivial homomorphisms. This implies that if a point p of any geometry G on S is mapped into the zero element by a homomorphism θ then the points on any non-trivial line which contains p are mapped by θ into the zero element. But then all the points contained in the equivalence class, induced by the non-trivial polygons of G, which contains p are identified with the zero element. Thus every homomorphism θ has to map all the points contained in a subset of the equivalence classes into the zero element. Conversely, to every subset of the equivalence classes there corresponds a homomorphism which identifies all the points in these equivalence classes with the zero element. Since every complete homomorphism θ on a complete point lattice is uniquely defined by the set of points which θ maps into the zero element we see that we have established a one-to-one order preserving mapping between the complete homomorphisms and the subsets of the equivalence classes.

The previous proof implies further the following result.

COROLLARY 3. *The lattice of complete homomorphisms of the lattice of subspaces of G on S is isomorphic to the Boolean algebra on the set of equivalence classes on S induced by the polygons of G.*

Finally we observe that any two complete homomorphisms on the lattice of subspaces of a geometry G permute.

So far we have characterized the complete homomorphisms of the lattices of subspaces of geometries. It remains an unsolved problem whether there are any incomplete homomorphisms in these lattices and if so how can these geometries be characterized. Furthermore, an interesting problem is to determine which geometries have complemented lattices of subspaces. Certainly one of the most interesting unsolved problems in lattice theory is Birkhoff's **(4)** problem number 48 which can be stated as follows: Is every finite lattice isomorphic to a sublattice of $L\,P_1(S)$ for some finite set S? So far we know by **(2)** and Theorem 3 that every finite lattice is isomorphic to a sublattice of $L\,P_n(S)$, S finite, $n > 2$.

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CONCERNING BINARY RELATIONS ON CONNECTED ORDERED SPACES

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1. Introduction. In a recent paper Mostert and Shields **(4)** showed that if a space homeomorphic to the non-negative real numbers is a certain type of topological semigroup, then the semigroup must be that of the non-negative real numbers with the usual multiplication. Somewhat earlier Faucett **(2)** showed that if a compact connected ordered space is a suitably restricted topological semigroup, then it must be both topologically and algebraically the same as the unit interval of real numbers with its usual multiplication.

In studying certain binary relations on topological spaces there have become known (see, in particular, Wallace **(5)** and the author **(3)**) a number of properties analogous to those possessed by topological semigroups. Because of these analogous properties between relations and semigroups the author was motivated by the general nature of the Faucett and Mostert-Shields results (that is, that the multiplication assumed turned out to be the same as the usual multiplication) to feel that certain relations on a connected ordered space should turn out to be the same as the orders whose order topologies are the topology on the space. (Eilenberg **(1)** showed, among other things, that a connected ordered space consisting of more than one point can be endowed with exactly two orders whose order topologies are the topology on the space, and these orders must be dual to each other.) The main result of this note is a characterization of these orders as reflexive transitive relations satisfying certain topological restrictions. As an immediate consequence of this characterization there is Faucett's result **(2**, Lemma 2**)** that if a compact connected ordered space S is a topological semigroup with zero, if the zero is an endpoint, and if each element of S has a left unit, then the binary relation on S,

$$\{(a, b) \in S \times S \mid a \in S b\},$$

is one of the two orders on S whose order topologies are the topology on S.

2. Preliminary definitions and results. *Throughout this paper it is assumed that X is a set consisting of more than a single element.* A set L will be called *a relation on X* provided $L \subset X \times X$ (the dual of L will be denoted by σL), and (X, L) will be called *an ordered set* provided L is reflexive, transitive, antisymmetric, and satisfies the requirement that if $x, y \in X$ then

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$(x, y) \in L$ or $(y, x) \in L$. If (X, L) is an ordered set, the usual terminology regarding lower bounds, upper bounds, infima, and suprema *with respect to* L of subsets of X will be used. If X is a topological space, then X will be called *an ordered space* provided there is such a relation L on X that (X, L) is an ordered set and the order topology induced by L *is* the topology on X.

If L is any relation on the topological space X, the following terminology and notation (in which, as throughout the paper, * is used to denote topological closure) will be employed:

(1) if $x \in X$ then $L(x) = \{y \in X \mid (y, x) \in L\}$;

(2) if $A \subset X$ then $L(A) = \bigcup\{L(a) \mid a \in A\}$;

(3) L will be called *continuous* (*monotone*) provided $L(A^*) \subset L(A)^*$ for each $A \subset X$ ($L(x)$ is connected for each $x \in X$);

(4) If $k \in X$ then k will be called *L-minimal* provided whenever $x \in X$ and $x \in L(k)$ then $k \in L(x)$; the set of L-minimal elements will be denoted by K_L;

(5) L will be called *closed below* (*closed above*) provided $L(x)$ ($\sigma L(x)$) is closed for each $x \in X$;

(6) if $B \subset X$ then B will be called an *L-ideal* provided $B \neq \phi$ and $L(B) \subset B$.

LEMMA. *Let (X, R) be a connected ordered space, and let L be a reflexive monotone continuous relation on X. If $x \in X - K_L$ then either $L(x) \subset R(x)$ or there exists $y \in R(X) - x$ such that $x \in L(y)$.*

Proof. Suppose $y \in R(X) - x$ implies $x \in X - L(y)$. If $y \in R(x) - x$ and if $L(y) \not\subset R(x) - x$, then there exists $z \in L(y)$ such that $x \in R(z)$; thus since L is reflexive and monotone $x \in \sigma R(y) \cap R(z) \subset L(y)$, a contradiction of the supposition. Therefore $y \in R(x) - x$ implies $L(y) \subset R(x) - x$, and it follows that $L(R(x) - x) = R(x) - x$. Hence from the reflexivity and continuity of L one has

$$L(x) \subset L(R(x)) = L((R(x) - x)^*) \subset L(R(x) - x)^* = (R(x) - x)^* = R(x),$$

which completes the proof.

COROLLARY. *Let (X, R) be a connected ordered space and let L be a reflexive transitive closed above monotone continuous relation on X. If $x \in X$ and if* $\inf \sigma L(x)$ *exists but does not belong to K_L, then $L(x) \subset R(x)$.*

Proof. Let $x \in X$ and suppose $L(x) \not\subset R(x)$ although $x_0 = \inf \sigma L(x)$ exists and $x_0 \in X - K_L$. By the lemma $x_0 \in R(x) - x$, and $x_0 \in \sigma L(x)$ since L is closed above. Thus $L(x_0) \not\subset R(x_0)$ so that again using the lemma, there exists $z \in R(x_0) - x_0$ such that $x_0 \in L(z)$. Because L is transitive it follows that $x \in L(z)$, that is, $z \in \sigma L(x)$; therefore $x_0 \neq \inf \sigma L(x)$, a contradiction. Hence it must be true that $L(x) \subset R(x)$.

3. Main result. It is well known that if L is a transitive closed below relation on the T_1 − space X and if A is a compact L-ideal, then $A \cap K_L \neq \phi$. This fact will be used in the proof of the following

THEOREM. *Let (X, R) be a connected ordered space, and let L be a relation on X. If L is reflexive, transitive, closed above and below, monotone, and continuous with $K_L = K_R$ or $K_L = K_{\sigma R}$, then $L = R$ or $L = \sigma R$ (not necessarily respectively). The converse is also true.*

Proof. The truth of the converse is obvious. The proof of the first statement is divided into two cases. It is assumed that $K_L = K_R$, for a completely dual proof holds in the dual case.

Case 1: $K_L \neq \phi$. Then K_R consists of a single endpoint of X, say e. It will be shown in this case that $L = R$, and for this it suffices to show that $L(x) = R(x)$ for each $x \in X$. Let $x \in X$. If $x = e$, then $e = K_R = K_L$ implies $L(e) = e = R(e)$. Suppose $x \neq e$, and let $x_0 = \inf \sigma L(x)$. Then $x_0 \in \sigma L(x)$, so that if $x_0 = e$ then $x \in L(e) = e$, a contradiction. Hence $x_0 \neq e$ and by the corollary $L(x) \subset R(x)$. Therefore since $L(x)$ is closed and $R(x)$ is compact, $L(x)$ is a compact L-ideal and thus $e = L(x) \cap K_L$. From the monotonicity of L it follows that $R(x) \subset L(x)$, and hence $L(x) = R(x)$.

Case 2: $K_R = \phi$.

(i) σL is monotone. To see this let $x \in X$ and suppose that $\sigma L(x)$ is not connected. Then there exists $c \in X - \sigma L(x)$ such that $A = \sigma L(x) \cap R(c)$ and $B = \sigma L(x) \cap \sigma R(c)$ are both non-void. Thus $a = \sup A$ and $b = \inf B$ exist, and it is easily seen that $a, b \in \sigma L(x)$ and

$$U = (\sigma R(a) \cap R(b)) - \{a, b\} \subset X - \sigma L(x).$$

Clearly $L(U)$ is connected so that $a, b \in X - U$ implies $L(U) \subset U$, that is, U is an L-ideal. But because L is continuous, U^* is a compact L-ideal and thus meets K_L, contrary to hypothesis. Consequently, $\sigma L(x)$ is connected and σL is monotone.

(ii) For each $x \in X$, either $L(x) = R(x)$ or $L(x) = \sigma R(x)$. Let $x \in X$. It suffices to show $L(x) \subset R(x)$ or $L(x) \subset \sigma R(x)$, for if $L(x) \subset R(x)$ but $L(x) \neq R(x)$, then $L(x)$ has a lower bound and is thus a compact L-ideal, implying $K_L \neq \phi$; and similarly, if $L(x) \subset \sigma R(x)$ then $L(x) = \sigma R(x)$. Suppose now that $L(x) \not\subset R(x)$. If $\sigma L(x)$ has a lower bound, then $\inf \sigma L(x)$ exists and by the corollary it follows that $L(x) \subset R(x)$, contrary to supposition. Therefore the monotonicity and reflexitivity of σL give $R(x) \subset \sigma L(x)$. If $\sigma L(x)$ also has no upper bound, then $X \subset \sigma L(x)$, implying $x \in K_L = \phi$. Let $x_0 = \sup \sigma L(x)$. (Note that $x_0 \neq \sup X$, for if $x_0 = \sup X$ then $X \subset \sigma L(x)$.) Then $\sigma L(x_0) \subset R(x_0)$ and hence $L(\sigma R(x_0) - x_0) \subset \sigma R(x_0) - x_0$, whence it follows that $L(x) \subset L(x_0) \subset L((\sigma R(x_0) - x_0)^*) \subset L(\sigma R(x_0) - x_0)^* \subset (\sigma R(x_0) - x_0)^* \subset \sigma R(x_0) \subset \sigma R(x)$.

(iii) If $y \in L(x) = R(x)$, then $L(y) = R(y)$. For $y \in L(x)$ implies $L(y) \subset L(x)$, and hence $L(y)$ has no lower bound since $L(y)$ is a closed L-ideal and $K_L = \phi$. Therefore (ii) implies $L(y) = R(y)$.

(iv) If $y \in L(x) = \sigma R(x)$, then $L(y) = \sigma R(y)$. The proof is similar to that of (iii).

(v) $L = R$ or $L = \sigma R$. Let $A = \{a \in X \mid L(a) = R(a)\}$ and let $B = \{b \in X \mid L(b) = \sigma R(b)\}$. Suppose $L \neq \sigma R$. Then from (ii) and (iii) it follows that A is connected and non-void. If A has no upper bound or if it has an upper bound which is also $\sup X$, then $A = X$ and $L = R$. If A has an upper bound which is not $\sup X$, then let $a_0 = \sup A$. It follows from (ii) and (iv) that B is connected and non-void; and using the continuity of L and the supposition $K_L = \phi$, it is not difficult to verify that $a_0 \in A \cap B$. But in order that $A \cap B \neq \phi$, it must be true that X consists of a single point, contrary to hypothesis. Hence $L = R$, and the proof of the theorem is complete.

4. Examples. A reflexive transitive relation satisfying all but one of the hypotheses of the theorem need be neither R nor σR. That this is indeed a fact is proved by the following set of examples in which X is, or is a subset of, the real numbers, and $R = \{(x, y) \in X \times X \mid x \leqq y\}$.

Example 1. Let X be the real numbers, and let $L = \{(x, y) \in X \times X \mid |x| \leqq |y|\}$. Then L is a reflexive transitive monotone continuous relation with closed graph (hence closed above and below), but $K_L = \{0\} \neq K_R = \phi = K_{\sigma R}$ and $R \neq L \neq \sigma R$.

Example 2. Let X and L be as in Example 1, and let $M = \sigma L$. Then M is a reflexive transitive continuous relation with closed graph, and $K_M = \phi = K_R$. However M is not monotone, and $R \neq M \neq \sigma R$.

Example 3. Let X be the real numbers, and let $L = \{(x, y) \in X \times X \mid x \leqq y \leqq 0\} \cup \{(x, y) \in X \times X \mid 0 \leqq y \leqq x\}$. Then L is a reflexive transitive monotone relation with closed graph, and $K_L = \phi = K_R$. However $R \neq L \neq \sigma R$ and L is not continuous.

Example 4. Let X be the set of real numbers x such that $0 < x \leqq 1$, and let

$$L_1 = \{(x, y) \in X \times X \mid x \leqq y < 1\} \cup \{(1, 1)\}.$$

If $L_2 = \sigma L_1$ then both L_1 and L_2 are reflexive transitive monotone continuous relations. Further, L_1 is closed below and L_2 is closed above, but L_1 is not closed above and L_2 is not closed below. Also

$$K_{L_1} = K_{L_2} = \{1\} = K_{\sigma R}$$

although $R \neq L_1 \neq \sigma R$ and $R \neq L_2 \neq \sigma R$.

5. Concerning a possible generalization. Let it be said (see Eilenberg **(1)** that a topological space X *can be ordered* provided there is a relation L on X such that (X, L) is an ordered set and the sets defined as open by the order topology are open in X under its original topology. It would be interesting to know if in the above theorem it is possible to replace "ordered space" by "space which can be ordered" and still have a true statement.

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ON A CLASS OF OPERATORS OCCURRING IN THE THEORY OF CHAINS OF INFINITE ORDER

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Introduction. Let T, E be two sets and $\mathfrak{T} \subset \mathfrak{P}(T)$,[1] $\mathfrak{E} \subset \mathfrak{P}(E)$ two tribes. For every $n \in N^*$ denote by E^n the product $E^{\{1,\ldots,n\}}$ and by $\mathfrak{E}^n$ the tribe $\mathfrak{E}^{\{1,\ldots,n\}}$. For every $x \in E$ let u_x be a mapping of T into T. For $x = (x_1, \ldots, x_n) \in E^n$ define $u_x = u_{x_n} \circ \ldots \circ u_{x_1}$ and suppose that $\{(t, x_1, \ldots, x_n) \mid u_{(x_1,\ldots,x_n)}(t) \in A\} \in \mathfrak{T} \oplus \mathfrak{E}^n$ for all $n \in N^*$ and $A \in \mathfrak{T}$.

Let $\mathfrak{M}$ be the Banach space of functions defined on T, real-valued, bounded and $\mathfrak{T}$-measurable with norm

$$\|f\| = \sup_{t \in T} |f(t)|.$$

For any sequence $S = (a_n)_{n \in N^*}$ of positive numbers, denote by $\mathfrak{M}_S$ the part of $\mathfrak{M}$ consisting of the functions f satisfying the inequality $|f(u_x(t_1)) - f(u_x(t_2))| \leqslant a_n$ for every $n \in N^*$, $x \in E^n$ and $t_1, t_2 \in T$.

Let p be a real-valued function defined on $T \times \mathfrak{E}$ having the properties:

(1) $0 \leqslant p(t, A) \leqslant p(t, E) = 1$ for $(t, A) \in T \times \mathfrak{E}$;

(2) $A \to p(t, A)$ is a completely additive measure for every $t \in T$;

(3) $t \to p(t, A)$ belongs, for every $A \in \mathfrak{E}$, to the same set $\mathfrak{M}_S$, where $S = (a_n)_{n \in N^*}$ is such that

$$\sum_{n \in N^*} a_n < \infty.$$

Define on $\mathfrak{M}$ the operator U by the equality

$$Uf(t) = \int_E p(t, dx) f(u_x(t)).$$

U is a linear operator of norm one which maps $\mathfrak{M}$ into $\mathfrak{M}$. Operators such as U occur in the study of certain stochastic models, especially in the theory of chains of infinite order (**1–4**; **6–10**; **12**; **14**; **15**). In this paper, under supplementary hypotheses, two ergodic properties of the sequence $(U^n)_{n \in N}$ will be proved. Under restrictive conditions it will be shown that the functions $t \to p(t, A)$ are conditional probabilities of a stationary mixing stochastic process (**8**, Theorem 6). Two other results, a non-homogeneous ergodic theorem and a central limit theorem, will also be given.

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[1]Some of the notations used in this paper are explained in paragraph 9 at the end of the paper.

1. For every $n \in N$ let $p_{1,n}$ be the function defined on $T \times \mathfrak{E}^n$ by

$$p_{1,n}(t, A) = \int_E p(t, dx_1) \int_E \cdots \int_E p(u_{(x_1,\ldots,x_{n-1})}(t), dx_n)\,\phi_A(x_1, \ldots, x_n), \qquad n > 1;$$

$$p_{1,n} = p, \qquad n = 1.$$

For any bounded sequence $C = (c_n)_{n \in N^*}$ write $\bar{C} = (\bar{c}_n)_{n \in N^*}$ where

$$\bar{c}_n = 4 \sum_{j \geq n} a_j + \sup_{j \geq n} c_j, \qquad n \in N^*.$$

The following three results will be needed below:

(i) for every $n \in N^*$, $p_{1,n}$ has the properties (1) to (3) if we replace $\mathfrak{E}$ by $\mathfrak{E}^n$ and $S = (a_k)_{k \in N^*}$ by

$$\Big(\sum_{k \leq j \leq k+n} a_j\Big)_{k \in N^*};$$

(ii) for every $n \in N^*$, $m \in N^*$ and $f \in \mathfrak{M}$, $t \in T$,

$$U^{n+m}f(t) = \int_{E^n} p_{1,n}(t, dx)\,U^m f(u_x(t));$$

(iii) if $f \in \mathfrak{M}_C$ where $C = (c_k)_{k \in N^*}$ then, for every $n \in N^*$, $U^n f \in \mathfrak{M}_{\|f\| \cdot \bar{C}}$.

2. Let us say that p satisfies condition (K) if there is on $\mathfrak{E}$ a completely additive measure μ, with value one on the whole space, and a constant $\lambda > 0$ such that $p(t, A) \geq \lambda\mu(A)$ for every $(t, A) \in T \times \mathfrak{E}$.

THEOREM 1. *If p satisfies condition (K) and $f \in \mathfrak{M}_C$ where $C = (c_n)_{n \in N^*}$ has the property*

$$\lim_{n \to \infty} c_n = 0$$

then there is a constant function $U^\infty f$ and a constant $0 < h = h_C < 1$ satisfying for every $n \in N^*$ the inequality

$$\|U^n f - U^\infty f\| \leq \|f\|^+ \inf_{1 \leq s \leq n} \big(\bar{c}_s/(1 - h) + 2h^{(n/s)-1}\big). \tag{4}$$

Choose an $r \in N^*$ such that

$$\sum_{j \geq r} a_j \leq \tfrac{1}{8}$$

and for every $n \in N^*$ let $\mu_n = \mu^1 \oplus \ldots \oplus \mu^n$ where $\mu^1 = \ldots = \mu^n = \mu$. For any $n \in N^*$ and $(t, A) \in T \times \mathfrak{E}^n$ let $\mu_n(t, A) = \mu_n(A)$ if $n \leq r$ and

$$\mu_n(t, A) = \int_{E^r} \mu_r(dx) \int_{E^{n-r}} p_{1,n-r}(u_{(x_1,\ldots,x_r)}(t), d(x_{r+1}, \ldots, x_n))\,\phi_A(x_1, \ldots, x_n)$$

if $n > r$. From the choice of r and property (i) it follows that $|\mu_n(t_1, A) - \mu_n(t_2, A)| \leq \frac{1}{4}$ for any $n \in N^*$, $t_1, t_2 \in T$ and $A \in \mathfrak{E}^n$. Using this inequality and condition (K) we obtain

$$p_{1,n}(t_1, A) \geq \lambda^r \mu_n(t_1, A) \geq \lambda^r \mu_n(t_2, A) - \tfrac{1}{4}\lambda^r. \tag{5}$$

For every $n \in N^*$, $t_1, t_2 \in T$ and $A \in \mathfrak{E}^n$, write $q_n(t_1, t_2; A) = p_{1,n}(t_1, A) - p_{1,n}(t_2, A)$. Let P, Q be two disjoint $\mathfrak{E}^n$-measurable sets whose union is E^n, such that $q_n(t_1, t_2; A) \geqslant 0$ if $A \subset P$, and $q_n(t_1, t_2; A) \leqslant 0$ if $A \subset Q$; we have then

$$B = q_n(t_1, t_2; P) = q_n(t_2, t_1; Q)$$

because $q_n(t_1, t_2; E^n) = 0$ (P, Q, and B depend on $n \in N^*$ and $t_1, t_2 \in T$). Using the inequality (5) we obtain

$$B \leqslant \inf\,(1 - p_{1,n}(t_2, P), 1 - p_{1,n}(t_1, Q)) \leqslant h = 1 - \tfrac{1}{4}\lambda^r. \tag{6}$$

Let us write the difference $U^n f(t_1) - U^n f(t_2)$ in the form

$$\int_{E^s} q_s(t_1, t_2; dx) U^{n-s} f(u_x(t_1)) + \int_{E^s} p_{1,s}(t_2, dx)(U^{n-s} f(u_x(t_1)) - U^{n-s} f(u_x(t_2))) \tag{7}$$

where $1 \leqslant s \leqslant n$. The second term in the sum (7) is less than or equal to $\|f\|^+ \tilde{c}_s$. If $B \neq 0$, the first term in the sum (7) can be written as

$$B\left(\int_P (q_s(t_1, t_2; dx)/B) U^{n-s} f(u_x(t_1)) - \int_Q (q_s(t_2, t_1; dx)/B) U^{n-s} f(u_x(t_1))\right)$$

and it follows from (6) that it is less than or equal to $h(\bar{f}^{n-s} - \underline{f}^{n-s})$. This inequality is obviously true if $B = 0$. Here, for every $k \in N$,

$$\bar{f}^k = \sup_{t \in T} U^k f(t), \quad \underline{f}^k = \inf_{t \in T} U^k f(t).$$

We obtain $\bar{f}^n - \underline{f}^n \leqslant \|f\|^+ \tilde{c}_s + h(\bar{f}^{n-s} - \underline{f}^{n-s})$. Hence for every integer $p \geqslant 1$ such that $ps \leqslant n$,

$$\bar{f}^n - \underline{f}^n \leqslant \|f\|^+ (1 + h + \ldots + h^{p-1}) \tilde{c}_s + h^p (\bar{f}^{n-ps} - \underline{f}^{n-ps}). \tag{8}$$

If we remark that the sequence $(\bar{f}^n)_{n \in N}$ is decreasing and that the sequence $(\underline{f}^n)_{n \in N}$ is increasing, then the existence of $U^\infty f$ and the inequality (4) follows from (8).

Remarks. 1° Denote by $\mathfrak{M}_1$ the union of the sets $\mathfrak{M}_C$ where $C = (c_n)_{n \in N^*}$ has the property

$$\lim_{n \to \infty} c_n = 0.$$

$\mathfrak{M}_1$ is a linear space and U^∞ is a linear form on $\mathfrak{M}_1$.

2° For every $n \in N^*$, $k \in N^*$ let us define the function $p^k{}_{1,n}$ on $T \times \mathfrak{E}^n$ by the equalities: $p^k{}_{1,n} = p_{1,n}$ if $k = 1$, and

$$p^k_{1,n}(t, A) = \int_E p(t, dx) p^{k-1}_{1,n}(u_x(t), A), \qquad (t, A) \in T \times \mathfrak{E}^n,$$

if $k > 1$. If we write $E^0 \times A = A$, then for every $n, k \in N^*$, $t \in T$ and $A \in \mathfrak{E}^n$, $p^k{}_{1,n}(t, A) = p_{1,n+k-1}(t, E^{k-1} \times A)$.

We deduce from Theorem 1 that for every $n \in N^*$ there is on $\mathfrak{E}^n$ a measure $p^{\infty}_{1,n}$, completely additive and with value one on the whole space, such that $(k > 1)$

$$|p_n^{k+1}(t, A) - p^{\infty}_{1,n}(A)| < \inf_{1<s<k} \left(6 \sum_{j>s} \frac{a_j}{1-h} + 2h^{(k/s)-1}\right). \tag{9}$$

3° If for any $n \in N^*$, $a_n = a^n$, $c_n = c^n$ where $0 < a, c < 1$, then the second member in the inequality (4) is dominated by $\|f\|^+ A_1 \exp(-q\sqrt{n})$ where $A_1 = A_1(a, c, \lambda)$ and $q = q(a, c, \lambda) > 0$.

3. Let us suppose in this paragraph that E is a finite set, and that for every $n \in N^*$ and $t \in T$ there is $x \in E^n$ and $t_n \in T$, such that $u_x(t_n) = t$. Under these hypotheses we can prove:

THEOREM 2. *For every $f \in \mathfrak{M}_1$ there is a function $U^{\infty} f \in \mathfrak{M}_1$ which satisfies the equality*

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{j=1}^{n} U^j f - U^{\infty} f \right\| = 0. \tag{10}$$

Let $t_0 \in T$ and denote by T_0 the set $\{u_x(t_0) \mid x \in E^n, n \in N^*\}$. As T_0 is denumerable, there is a strictly increasing subsequence of N^*, $(n_j)_{j\in N^*}$ such that the sequence

$$\left(\frac{1}{n_j} \sum_{i=1}^{n_j} U^i f(t)\right)_{1<j<\infty}$$

is convergent for every $t \in T_0$. Using property (iii) we deduce that there exists a function $U^{\infty} f \in \mathfrak{M}_1$ satisfying the equality

$$\lim_{j\to\infty} \left\| \frac{1}{n_j} \sum_{i=1}^{n_j} U^i f - U^{\infty} f \right\| = 0.$$

As $\|U^j\| = 1$ for every $j \in N$, the mean ergodic theorem of Yosida and Kakutani **(16)** implies (10).

U^{∞} can be extended uniquely to the closure of $\mathfrak{M}_1$ in $\mathfrak{M}$, $\bar{\mathfrak{M}}_1$, and (10) remains valid for $f \in \bar{\mathfrak{M}}_1$.

4. For any $n \in Z$ let p_n be a real-valued function defined on $T \times \mathfrak{E}$, having the properties:

(11) $0 < p_n(t, A) < p_n(t, E) = 1$ for $(t, A) \in T \times \mathfrak{E}$;

(12) $A \to p_n(t, A)$ is a completely additive measure for every $t \in T$;

(13) $t \to p_n(t, A)$ belongs, for every $n \in Z$ and $A \in \mathfrak{E}$, to the same set $\mathfrak{M}_S$ where $S = (a_k)_{k\in N^*}$ is such that

$$\sum_{n\in N^*} a_n < \infty.$$

For every $n \in Z$ define on $\mathfrak{M}$ the operator $U^{n-1,n}$ by the equality

$$U^{n-1,n} f(t) = \int_E p_n(t, dx) f(u_x(t)).$$

For $(n, m) \in \mathfrak{Z} = \{(n, m) | n \in Z, m \in Z, n \leq m\}$ write $U^{n,m} = U^{n,n+1} \circ \ldots \circ U^{m-1,m}$ if $n < m$ and $U^{n,m} = I$ if $n = m$. Let us say that the family $(p_n)_{n \in Z}$ satisfies condition (K) if there is on $\mathfrak{E}$ a family $(\mu_n)_{n \in Z}$ of completely additive measures, having value one on the whole space, and a constant $\lambda > 0$ such that $p_n(t, A) \geq \lambda \mu_n(A)$ for every $(t, A) \in T \times \mathfrak{E}$ and $n \in Z$. By an argument similar to the one used in the proof of Theorem 1, but somewhat more involved, we can obtain:

THEOREM 3. *If the family $(p_n)_{n \in Z}$ satisfies condition (K) and $f \in M_C$, then there is a constant $0 < h = h_C < 1$ satisfying for every $(n, m) \in \mathfrak{Z}$, $n < m$, and t_1, t_2 the inequality*

$$(14) \quad |U^{n,m}f(t_1) - U^{n,m}f(t_2)| \leq \|f\|^+ \inf_{1 \leq s \leq m-n} (\bar{c}_s/(1 - h) + h^{((m-n)/s)-1}).$$

Here C does not necessarily satisfy any supplementary condition.

If the sequence $C = (c_n)_{n \in N^*}$ is such that

$$\lim_{n \to \infty} c_n = 0$$

then it follows from (14) that

$$\lim_{m-n \to \infty} (U^{n,m}f(t_1) - U^{n,m}f(t_2)) = 0$$

uniformly with respect to $t_1, t_2 \in T$ and f in a given bounded part of $\mathfrak{M}_C$.

5. Suppose now that:

(15) E is metric complete and separable and $\mathfrak{E}$ is the tribe of Borel parts of E;

(16) $T = E^{-N}$ and $\mathfrak{T} = \mathfrak{E}^{-N}$ where $-N = \{\ldots, -1, 0\}$;

(17) u_x, $x \in E$, is defined on T by: $u_x((\ldots, x_{-1}, x_0)) = (\ldots, x_0, x)$.

For every $(n, m) \in \mathfrak{Z}$ define the function $p^\infty_{n,m}$ on $\mathfrak{E}^{(n, \ldots, m)}$ by the equality (we identify $\mathfrak{E}^{m-n+1}$ with $\mathfrak{E}^{(n, \ldots, m)}$): $p^\infty_{n,m}(A) = p^\infty_{1,m-n+1}(A)$, and for every $(n, m) \in \mathfrak{Z}$ and $r \in N^*$ define the function $p^r_{n,m}$ on $T \times \mathfrak{E}^{(n, \ldots, m)}$ by: $p_{n,m}{}^r(t, A) = p_{1,m-n+1}{}^r(t, A)$.

THEOREM 4. *If p satisfies condition (K), then there is one and only one stochastic process $(E^Z, \mathfrak{E}^Z, p^Z)$ such that the equality*

$$(18) \qquad p^Z\{pr^{-1}_{n+1}(A) | pr_{(\ldots, n)}(\omega) = t\} = p(t, A)$$

is satisfied almost everywhere for any $n \in Z$ and $A \in \mathfrak{E}$. The stochastic process $(E^Z, \mathfrak{E}^Z, p^Z)$ is stationary and strongly mixing.

Let us remark that if (n, m), $(n', m') \in \mathfrak{Z}$ and

$$pr^{-1}_{(n', \ldots, m')}(B) = pr^{-1}_{(n, \ldots, m)}(A) \in \mathfrak{E}^Z$$

then

$$p^\infty_{n,m}(A) = p^\infty_{n',m'}(B).$$

Hence there is one and only one stochastic process $(E^Z, \mathfrak{E}^Z, p^Z)$ such that $p^Z(pr^{-1}{}_{\{n,\ldots,m\}}(A)) = p^\infty{}_{n,m}(A)$ for $(n, m) \in \mathfrak{Z}$ and $pr^{-1}{}_{\{n,\ldots,m\}}(A) \in \mathfrak{E}^Z$. Hypothesis (15) is used here only. We leave to the reader to verify that the process is stationary.

For every $(s, m) \in \mathfrak{Z}$ let us define the function $\bar{p}_{s,m}$ on $E^{\{s,\ldots,m\}} \times \mathfrak{E}$ by the equality $\bar{p}_{s,m}(x, A) = p(u_x(t_0), A)$ where $t_0 \in T$ is a fixed element. For $n < s < m$, $A \in \mathfrak{E}$ and $M \in \mathfrak{E}^{\{n,\ldots,m\}}$ we have then

$$\int p^Z(d\omega)p(pr_{\{\ldots,m\}}(\omega), A) = \theta_1 a_{m-s+1} + \int p^Z(d\omega)\bar{p}_{s,m}(pr_{\{s,\ldots,m\}}(\omega), A)$$

$$= \theta_1 a_{m-s+1} + \int_M p_{n,m}(dx)\bar{p}_{s,m}(x_{s,m}, A) = \theta_1 a_{m-s+1} + \lim_{r\to\infty}\int_M p^{r+1}_{n,m}(t, dx)\bar{p}_{s,m}(x_{s,m}, A)$$

where the first two integrals are taken over $pr^{-1}{}_{\{n,\ldots,m\}}(M)$, $|\theta_1| < 1$, and $x_{s,m} = (x_s, \ldots, x_m)$ if $x = (x_{n'}, \ldots, x_m)$ and $n' < s < m$. For any $r \in Z$

$$\int_M p^{r+1}_{n,m}(t, dx)\bar{p}_{s,m}(x_{s,m}, A) = \int_{M(r)} p_{n-r,m}(t, dx)\bar{p}_{s,m}(x_{s,m}, A)$$

$$= \theta_2 a_{m-s+1} + \int_{M(r)} p_{n-r,m}(t, dx)p(u_x(t), A) = \theta_2 a_{m-s+1} + p^{r+1}_{n,m+1}(t, M \times A)$$

where $M(r) = E^{\{n-r,\ldots,n-1\}} \times M$, and $|\theta_2| < 1$. It follows that

$$\int p^Z(d\omega)p(pr_{\{\ldots,m\}}(\omega), A) = \theta_3 a_{m-s+1} + \lim_{r\to\infty} p^r_{n,m+1}(t, M \times A)$$

$$= \theta_3 a_{m-s+1} + p^\infty_{n,m+1}(M \times A) = \theta_3 a_{m-s+1} + p^Z(pr^{-1}_{\{n,\ldots,m\}}(M) \cap pr^{-1}_{m+1}(A)),$$

the integral being taken over $pr^{-1}{}_{\{n,\ldots,m\}}(M)$, and $|\theta_3| < 2$. But

$$pr^{-1}_{\{n,\ldots,m\}}(M) = pr^{-1}_{\{n',\ldots,m\}}(E^{\{n',\ldots,n-1\}} \times M) \qquad \text{if } n' < n;$$

hence s can be allowed to tend to $-\infty$ in the above formula. It follows that (18) is satisfied almost everywhere.

Suppose now that $(E^Z, \mathfrak{E}^Z, \bar{p})$ is a stochastic process such that the equality $\bar{p}\{pr^{-1}{}_{n+1}(A)|pr^{-1}{}_{\{\ldots,n\}}(\omega) = t\} = p(t, A)$ is satisfied almost everywhere for any $n \in Z$ and $A \in \mathfrak{E}$. Then for $(n, m) \in \mathfrak{Z}$, $r > 1$ and $A \in \mathfrak{E}^{\{n,\ldots,m\}}$ we have

$$\int_{E^Z} \bar{p}(d\omega)p_{n-r,m}(pr_{\{\ldots,n-r-1\}}(\omega), E^{\{n-r,\ldots,n-1\}} \times A) = \bar{p}(pr^{-1}_{\{n,\ldots,m\}}(A)).$$

If we let r tend to ∞ and use (9) we obtain $\bar{p}(pr^{-1}{}_{\{n,\ldots,m\}}(A)) = p^Z(pr^{-1}{}_{\{n,\ldots,m\}}(A))$; therefore $\bar{p} = p^Z$.

It remains to prove that $(E^Z, \mathfrak{E}^Z, p^Z)$ is strongly mixing. For this it is sufficient to show that

$$\lim_{n\to\infty} p^Z(\tau^n(A) \cap B) = p^Z(A)p^Z(B)$$

for every $A = pr^{-1}{}_{\{s,\ldots,z\}}(A_1) \in \mathfrak{E}^Z$ and $B = pr^{-1}{}_{\{s,\ldots,z\}}(B_1) \in \mathfrak{E}^Z$. But if we remark that $\tau^n(A) = pr^{-1}{}_{\{s-n,\ldots,z-n\}}(A_1)$ we obtain that

$$\lim_{n\to\infty} p^Z(\tau^n(A) \cap B) = \lim_{n\to\infty}\int_{\tau^n(A)} p^Z(d\omega)p_{z-n+1,z}(pr_{\{\ldots,z-n\}}(\omega), E^{\{z-n+1,\ldots,s-1\}} \times B_1)$$

is equal to $p^{\mathbf{Z}}(A)p^{\mathbf{Z}}(B)$. Hence the process is strongly mixing and so the theorem is proved.

6. Let us suppose that the conditions under which Theorem 4 has been proved are satisfied and also that

$$\sum_{n\geq 1} n\Big(\sum_{j>\sqrt{n}} a_j\Big)^{\frac{1}{2}} < \infty.$$

Let f be a function, real-valued, $\mathfrak{E}^r$-measurable, defined on E^r. For every $n \in N^*$ write $f_n = f \circ pr_{\{n,\ldots,n+r-1\}}$ (we identify $E^{\{n,\ldots,n+r-1\}}$ with E^r and $\mathfrak{E}^{\{n,\ldots,n+r-1\}}$ with $\mathfrak{E}^r$). We have then:

THEOREM 5. *Suppose* $E(f_1) = 0$ *and* $E(|f|^{\alpha}) < \infty$ *for an* $\alpha > 2$. *Then*:

(j) *the series*

$$D = E(f_1^2) + 2\sum_{i\in N^*} E(f_1 f_{1+i})$$

converges absolutely and, for $n \to \infty$, $E((f_1 + \ldots + f_n)^2/n) = D + 0\,(1/n)$;

(jj) *if* $D \neq 0$ *we have uniformly in* a

$$\lim_{n\to\infty} p^{\mathbf{Z}}\left(\frac{(f_1 + \ldots + f_n)}{\sqrt{n}} < a\right) = (1/(2\pi D)^{\frac{1}{2}})\int_{-\infty}^{a} \exp(-t^2/2D)dt. \tag{19}$$

The expectations are calculated with respect to the measure $p^{\mathbf{Z}}$. Once the existence of the stationary process $(E^{\mathbf{Z}}, \mathfrak{E}^{\mathbf{Z}}, p^{\mathbf{Z}})$ is established, the theorem can be obtained by the method used by Doob to prove the central limit theorem for Markoff process (**5**, 221–32). We shall not give details here.

7. The first explicit and systematic study of chains of infinite order was made in (**15**). The transition probabilities of the chains studied in (**15**, 6–11), as well as the transition probabilities of chains of type (A) introduced in (**4**) and of chains of type (B) introduced in (**2**), (**3**), and (**4**) satisfy conditions (1)–(3). It follows that the theorems A and D (**2**), the ergodic theorem proved in (**15**, 6–11), the formulas given in (**4**, 139) (the evaluations are slightly different from those given by formula (9)), and the theorem I_3, (**6**, 423–6) (in the case when $|\phi_i| < 1$ for every i) are particular cases of Theorem 1. The convergence property of the transition probabilities, established in Theorem II, (**4**, 137) is also a consequence of Theorem 1. For chains of type (A) some stronger results, expressed by formula (22), are valid. Under different conditions the C_1 convergence of the sequence $(p^r{}_{1,n})_{r\in N^*}$ has been proved in (**8**, Theorem 6,c). This result is not contained in, nor does it contain the one proved in Theorem 2. If E is a finite set, results similar to Theorem 4 are given in (**8**), under weaker conditions. Various kinds of central limit theorems, having points of contact with Theorem 5 have been given in (**2**; **3**; **7**; **14**).

8. Suppose now that T is a compact metric space, $\mathfrak{T}$ the tribe of Borel parts of T and p a real-valued function defined on $T \times \mathfrak{E}$ having the properties (1), (2) and:

$$|p(t_1, A) - p(t_2, A)| \leqslant Kd(t_1, t_2) \tag{20}$$

for every $A \in \mathfrak{E}$ and $t_1, t_2 \in T$. Suppose further that there is a constant $0 < r < 1$ such that

$$d(u_x(t_1), u_x(t_2)) \leqslant rd(t_1, t_2)$$

for any $x \in E$ and $t_1, t_2 \in T$. It follows then that p satisfies condition (3) if we take $a_n = Mr^n$, where $M = K \times$ diameter of T, for every $n \in N^*$.

Denote by $\mathfrak{CL}$ the Banach space of complex-valued functions defined on T satisfying the Lipschitz condition, the norm being given by $\|f\|_1 = \|f\| + m(f)$ where $\|f\| = \sup_{t \in T} |f(t)|$ and

$$m(f) = \sup_{t_1 \neq t_2} \frac{|f(t_1) - f(t_2)|}{d(t_1, t_2)}.$$

We remark that the real and the imaginary part of every function $f \in \mathfrak{CL}$ belongs to $\mathfrak{M}_{m(f)S}$ where $S = (a_n)_{n \in N}$; in particular they belong to $\mathfrak{M}_1$. Define the operator U on $\mathfrak{CL}$ by

$$Uf(t) = \int_E p(t, dx) f(u_x(t)). \tag{21}$$

Then **(12; 13)** U maps CL into CL, U is quasi-compact, the sequence $(\|U^n\|_1)_{n \in N}$ is bounded and 1 is a characteristic value of U.

It follows from Theorem 1 that if p satisfies condition (K), then for every $f \in \mathfrak{CL}$ the sequence $(U^n f)_{n \in N}$ converges uniformly to a constant function $U^\infty f$. But this result implies that the only characteristic value of U of modulus one is 1 and that this characteristic value is simple. Using the properties of U mentioned above we deduce that there are two constants M and $\nu > 0$ satisfying the inequality

$$\|U^n - U^\infty\|_1 \leqslant \frac{M}{(1 + \nu)^n} \tag{22}$$

for every $n \in N^*$.

The operator U can be defined by formula (21) also for $f \in \mathfrak{C}$. For every $n \in N^*$, $\|U^n\| = 1$. As $\mathfrak{CL}$ is dense in $\mathfrak{C}$, it follows that U^∞ can be extended uniquely to $\mathfrak{C}$, and that

$$\lim_{n \to \infty} \|U^n f - U^\infty f\| = 0 \qquad \text{for every } f \in \mathfrak{C}. \tag{23}$$

This proposition contains some results proved in **(9, §6)**.

Let us make one more remark. Suppose, in addition, that:

(α) E is a topological space and $\mathfrak{E}$ contains the open sets;

(β) the mapping $x \to u_x(t)$ is continuous for every $t \in T$;

(γ) for every open set $V \subset T$ there is $n(V) \in N^*$ and $x(V) \in E^n(V)$ such that $u_{x(V)}(t) \in V$ for every $t \in T$.

The conditions (α)–(γ) are satisfied in the case of chains of type (A) **(3; 12)**. If p satisfies condition (K): $p(t, A) \geqslant \lambda\mu(A)$ for every $(t, A) \in T \times \mathfrak{E}$ where $\lambda > 0$ and $\mu(A) > 0$ for every open set A, then U is strongly positive with respect to the cone $\{f \mid f \in \mathfrak{C}\mathfrak{L}, f \geqslant 0\}$. We can then obtain[2] (22), more directly, using a slight modification of Theorem 6.3, (a) and (c), 70–3, **(11)**.

9. We shall explain in this paragraph some of the notations used in the paper.

For each set X, $\mathfrak{P}(X)$ is the set of parts of X. $N = \{0, 1, \dots\}$, $N^* = \{1, 2, \dots\}$, $Z = \{\dots, -1, 0, 1, \dots\}$. A part $\mathfrak{T} \subset \mathfrak{P}(X)$ is a tribe if $\mathfrak{T} \ni X$, $\mathfrak{T} \ni X - A$ if $\mathfrak{T} \ni A$ and $\mathfrak{T} \ni \bigcup_{n \in N} A_n$ if $\mathfrak{T} \ni A_n$ for every $n \in N$.

For every $I \subset Z$ we denote by E^I the product

$$\prod_{j \in I} E_j$$

where $E_j = E$ for $j \in I$. By $\mathfrak{E}^I \subset \mathfrak{P}(E^I)$ we denote the smallest tribe containing the sets of the form

$$\prod_{j \in I} A_j$$

where $A_j \in \mathfrak{E}$ for $j \in I$.

For every real number α we write $\alpha^+ = \sup(\alpha, 1)$. If α is a real number and $\bar{C} = (\bar{c}_n)_{n \in N^*}$, then $\alpha\bar{C} = (\alpha\bar{c}_n)_{n \in N^*}$.

τ is the mapping of E^Z into E^Z defined by the equality: $\tau((x_n)_{n \in Z}) = (x_{n+1})_{n \in Z}$.

$\mathfrak{C}$ is the Banach space of continuous complex-valued functions defined on T with the norm $\|f\| = \sup_{t \in T} |f(t)|$.

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[2]The details were given recently in the Functional Analysis Seminar.

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TYPICALLY-REAL FUNCTIONS

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1. Introduction. A function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

a_n real, is called typically-real of order one in the closed region $|z| \leq R$ if it satisfies the following conditions **(6)**.

(1) $f(z)$ is regular in $|z| \leq R$.

(2) $\mathscr{I}\{f(z)\} > 0$ if and only if $\mathscr{I}\{z\} > 0$.

The same function is called typically-real of order p, p a positive integer greater than one, if it satisfies condition (1) above and in addition the following condition **(4; 5)**:

(2′) there exists a constant ρ, $0 < \rho < R$, such that on every circle $|z| = r$, $\rho < r < R$, $\mathscr{I}\{f(z)\}$ changes sign exactly $2p$ times.

We shall denote the class of functions which are typically-real of order p in the open disc $|z| < R$ by $T_p^*(R)$ while those which are typically-real of order p in the closed disc $|z| \leq R$ will be denoted by $T_p(R)$.

In the proofs which follow we assume that all functions belong to $T_p(1)$. The results will remain valid for the larger class $T_p^*(1)$ by noting that if $f(z) \in T_p^*(1)$ then $f(rz)/r \in T_p(1)$ for all $\rho < r < 1$ **(2)**.

The problem to be considered in this paper is that of determining a positive expresssion R_p depending upon the first p coefficients of $f(z)$ with the following property:

$$f(z) \in T_p(1) \Rightarrow f(z) \in T_1(R_p).$$

In §§ 3 and 4 we will develop a recursion relationship for R_p, $p = 1, 2, 3, \ldots$, and in § 6 we will show that our definition of R_p is sharp for the class of functions, $\bigcup_p T_p(1)$ in the sense that for $p = 2$ it is the best possible bound.

2. A Representation theorem. We shall first develop an integral representation for functions of class $T_p(1)$, $p > 1$.

THEOREM 2.1. *If there exists a function R_{p-1} $(c_2, c_3, \ldots, c_{p-1}) > 0$ with the property that for any function*

$$g(z) = z + \sum_{n=2}^{\infty} c_n z^n \in T_{p-1}(1)$$

we have $g(z) \in T_1(R_{p-1})$, then given an arbitrary function

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$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in T_p(1)$$

we can write

$$f(R_{p-1}\omega) = \frac{2}{\pi}\int_0^{\pi} P(\omega, \nu, \phi)\, d\alpha(\phi), \qquad |\omega| < 1 \tag{2.1}$$

where

$$P(\omega, \nu, \phi) = \frac{\omega^3 + (R_{p-1}b_1 - 2\cos\phi)\omega^2 + \omega}{(1 - 2\omega\cos\phi + \omega^2)(1 - 2R_{p-1}\omega\cos\nu + R_{p-1}^2\omega^2)},$$

$b_1 = a_2 - 2\cos\nu \neq 0$, $0 < \nu < \pi$, $c_1, \dots, c_{p-1}$ are given by (2.2), *and* $d\alpha(\phi) > 0$ for $0 < \phi < \pi$.

Proof. Since $f(z) \in T_p(1)$ we have from **(3)** that

$$g(z) = \frac{1 - 2z\cos\nu + z^2}{b_1 z} f(z) - \frac{1}{b_1} = z + \sum_{n=2}^{\infty} c_n z^n \in T_{p-1}(1), \tag{2.2}$$

where ν is chosen subject to the following conditions:

(1) $0 < \nu < \pi$.

(2) $\mathscr{I}\{f(z)\}$ changes sign at $z = e^{i\nu}$.

(3) $b_1 = a_2 - 2\cos\nu \neq 0$.

(4) $b_1 > 0$ if $p = 2$.

It follows then from the hypotheses of Theorem 2.1 that $g(z) \in T_1(R_{p-1})$. It should also be noted that from (2.2) it follows that the R_{p-1} of Theorem 2.1 is a function of $a_2, \dots, a_p$ and ν.

Let us now compute the coefficients c_n of $g(z)$ by integrating over the path C: $|z| = R_{p-1}$. This yields

$$\begin{aligned} c_n &= \frac{1}{2\pi i}\int_C \frac{g(z)}{z^{n+1}}\, dz \\ &= \frac{1}{2\pi R_{p-1}^n}\int_0^{2\pi} g(R_{p-1}e^{i\phi})e^{-in\phi}\, d\phi \qquad \text{where } z = \rho e^{i\phi}. \end{aligned} \tag{2.3}$$

Adding to (2.3) the expression

$$\frac{1}{2\pi R_{p-1}^n}\int_0^{2\pi} g(R_{p-1}e^{i\phi})e^{in\phi}\, d\phi = 0$$

we obtain

$$c_n = \frac{-i}{\pi R_{p-1}^n}\int_0^{2\pi} g(R_{p-1}e^{i\phi}) \sin n\phi\, d\phi. \tag{2.4}$$

If we now let $g(R_{p-1}e^{i\phi}) = u(R_{p-1}e^{i\phi}) + iv(R_{p-1}e^{i\phi})$ we have, since the c_n are real,

$$c_n = \frac{1}{\pi R_{p-1}^n}\int_0^{2\pi} v(R_{p-1}e^{i\phi}) \sin n\phi\, d\phi. \tag{2.5}$$

Since, however, $v(R_{p-1}e^{i\phi}) > 0$ for all $0 < \phi < \pi$ and since $v(R_{p-1}e^{i\phi}) = -v(R_{p-1}e^{-i\phi})$ we have

$$c_n = \frac{2}{\pi R_{p-1}^n} \int_0^\pi v(R_{p-1}e^{i\phi}) \sin n\phi \, d\phi$$

where

$$\frac{2}{\pi R_{p-1}} \int_\pi^0 v(R_{p-1}e^{i\phi}) \sin \phi \, d\phi = 1. \tag{2.6}$$

Thus

$$\begin{aligned} g(z) &= \sum_{n=1}^{\infty} \left[\frac{2}{\pi R_{p-1}^n} \int_0^\pi v(R_{p-1}e^{i\phi}) \sin \phi \, d\phi \right] z^n. \\ &= \frac{2}{\pi} \int_0^\pi \left[v(R_{p-1}e^{i\phi}) \sin \phi \sum_{n=1}^{\infty} \frac{\sin n\phi}{\sin \phi} \left(\frac{z}{R_{p-1}} \right)^n \right] d\phi \\ &= \frac{2}{\pi} \int_0^\pi \frac{R_{p-1} z \, v(R_{p-1}e^{i\phi}) \sin \phi \, d\phi}{R_{p-1}^2 - 2R_{p-1} z \cos \phi + z^2}, \qquad |z| < R_{p-1}. \end{aligned}$$

Thus

$$f(z) = \frac{2}{\pi} \int_0^\pi \frac{b_1 R_{p-1} z^2 \, d\alpha(\phi)}{(R_{p-1}^2 - 2R_{p-1} z \cos \phi + z^2)(1 - 2z \cos \nu + z^2)} + \frac{z}{1 - 2z \cos \nu + z^2} \tag{2.7}$$

where $d\alpha(\phi) \equiv v(R_{p-1}e^{i\phi}) \sin \phi > 0$ for all $0 < \phi < \pi$.

Using (2.6) we can rewrite (2.7) in the form

$$f(z) = \frac{2}{\pi R_{p-1}} \int_0^\pi \frac{[z^3 + R_{p-1}(R_{p-1}b_1 - 2 \cos \phi) z^2 + R_{p-1}{}^2 z]}{(R_{p-1}^2 - 2R_{p-1} z \cos \phi + z^2)(1 - 2z \cos \nu + z^2)} \, d\alpha(\phi), \qquad |z| < R_{p-1}. \tag{2.8}$$

The transformation of variable $z = R_{p-1}\omega$ now gives (2.1).

From (2.1) it follows that

$$\mathscr{J}\{P\} = 4r^3(1 - r^2 R_{p-1})^2 \, . \, D^{-2} \, . \, \sin \theta \, (\cos^2\theta + B \cos \theta + C) \tag{2.9}$$

where $\omega = re^{i\theta}$,

$$\begin{aligned} D^2 &\equiv D^2(a_2, \ldots, a_p; r, \nu, \phi) \\ &= |(1 - 2\omega \cos \phi + \omega^2)\,(1 - 2R_{p-1}\omega \cos \nu + R_{p-1}{}^2\omega^2)|^2 > 0 \end{aligned} \tag{2.10}$$

for all $R_{p-1} < 1$, $|\omega| < 1$,

$$\begin{aligned} B &\equiv B(a_2, \ldots, a_p; r, \nu, \phi) \\ &= \frac{K(1 + r^2)(1 - r^2 R_{p-1}^2) - b_1 r^2 R_{p-1}(1 - R_{p-1}^2)}{2r(1 - r^2 R_{p-1}^2)}. \end{aligned} \tag{2.11}$$

$$C \equiv C(a_2, \ldots, a_p; r, \nu, \phi) = \tag{2.12}$$

$$\frac{\begin{array}{l} - R_{p-1}^2 r^6 + [K R_{p-1}^3 b_1 - K^2 R_{p-1}^2 + 2R_{p-1}^2 + 2b_1 R_{p-1}^2 \cos \nu + 1] r^4 \\ \qquad - [K b_1 R_{p-1} - K^2 + R_{p-1}^2 + 2b_1 R_{p-1}^2 \cos \nu + 2] r^2 + 1 \end{array}}{4r^2(1 - r^2 R_{p-1}^2)}$$

and

(2.13) $$K \equiv K(a_2, \ldots, a_p; r, \nu, \phi) = R_{p-1}b_1 - 2\cos\phi.$$

3. Definition of $\bar{R}_n(a_2, \ldots, a_p)$, $p > 1$. Given any function of the class $T_p(1)$ consider the equation

(3.1) $$\frac{\partial P}{\partial \omega} \equiv P'(\omega) = 0 \qquad (a_2, \ldots, a_p \text{ fixed.})$$

Find all of the real roots $\omega_1(\nu, \phi)$, $\omega_2(\nu, \phi), \ldots, \omega_k(\nu, \phi)$, $1 \leqslant k \leqslant 6$ of (3.1). We then define

(3.2) $$\bar{R}_p \equiv \bar{R}_p(a_2, \ldots, a_p) = \min_{\nu,\phi} |\omega_i(\nu, \phi)|, \qquad i = 1, \ldots, k,$$

where the minimum is taken over all ν, ϕ satisfying $0 \leqslant \phi \leqslant \pi$, $0 \leqslant \nu \leqslant \pi$. We note here that $P'(0) = 1$,

(3.3) $$P'(1) = \frac{[R_{p-1}b_1 + 2(1 - \cos\phi)](R_{p-1}^2 - 4R_{p-1}\cos\nu + 3)}{2(1 - \cos\phi)(1 - 2R_{p-1}\cos\nu + R_{p-1}^2)^2},$$

and

(3.4) $$P'(-1) = \frac{[-R_{p-1}b_1 + 2(1 - \cos\phi)](1 - R_{p-1}^2)}{2(1 + \cos\phi)(1 + 2R_{p-1}\cos\nu + R_{p-1}^2)^2}.$$

It is clear then that for $0 < R_{p-1} < 1$ if $R_1b_1 > 0$ then there exists a ϕ such that $P'(-1) < 0$ while if $R_1b_1 < 0$ then there exists a ν and ϕ such that $P'(1) < 0$. Thus if $0 < R_{p-1} < 1$

(3.5) $$\bar{R}_p < 1.$$

4. The main theorem. From our definition of P in (2.1) and from (2.9) and (2.10) it is clear that any variation in the sign of $\mathscr{I}\{P\}$ for $0 < \theta < \pi$ must result from a variation in the sign of the factor $(\cos^2\theta + B\cos\theta + C)$. Thus, for any $0 \leqq r \leqq 1$ the functions P must be members of one of the three classes $T_i(r)$, $i = 1, 2, 3$. It should also be noted here that if for a particular value of r a function belongs to $T_1(r)$, then that function belongs to $T_1(r)$ for all smaller values of r.

Next we note that if for $0 < r_1 < r_2 < 1$ and fixed $a_2, a_3, \ldots, a_p, \nu$, and ϕ we have $P \in T_2(r_2)$, $P \in T_1(r_1)$ and $P \notin T_3(r)$ for any r satisfying $r_1 < r < r_2$ then there must exist an r satisfying $r_1 \leqslant r \leqslant r_2$ for which either $P'(r) = 0$ or $P'(-r) = 0$. This follows directly from the relation $\mathscr{I}\{P(\omega)\} = \mathscr{I}\{P(\bar{\omega})\}$, $|\omega| \leqq 1$, and the analyticity of all the P in $|\omega| < 1$.

From the definition of $\bar{R}_p$ in § 3 and from the preceding paragraph it is clear that for fixed $a_2, a_3, \ldots, a_p$ and $r < \bar{R}_p$, no function P can change directly from the class $T_2(r)$ to $T_1(r)$.

If, then, we are able to show that for any choice of $a_2, a_3, \ldots, a_p, \nu, \phi$ there exists no r, $0 < r < \bar{R}_p$, for which we have both $B^2 - 4 \leqslant 0$ and $B^2 - 4C \geqslant 0$ we will have shown that for $r < \bar{R}_p$ we cannot have $P \in T_3(r)$ and with the result of paragraph (4.3) will have established the

Main Theorem. *If $f(z) \in T_p(1)$, p a positive integer greater than* 1, *then $f(z) \in T_1(r)$ for all r satisfying $0 < r < R_p\ (a_2, \ldots, a_p) = R_{p-1}\tilde{R}_p$, where $R_1 = 1$ and $\tilde{R}_p$ is as defined in* § 2.

In the proofs which follow in this section we will assume that $f(z) \in T_p(1)$, $p > 2$, and that $R_{p-1} < 1$. The case $p = 2$ will be treated separately in § 5. In § 5 we also show that $R_2 < 1$. This, then, justifies the assumption $R_{p-1} < 1$, $p > 2$.

The proof of the Main Theorem will depend upon four lemmas. In the proof of these we will fix $a_2, \ldots, a_p, \nu$ in the expressions $B^2 - 4 = 0$ and $B^2 - 4C = 0$ and plot r against K. The lemmas will be used to prove that the general geometric configuration is that of Figure 1.

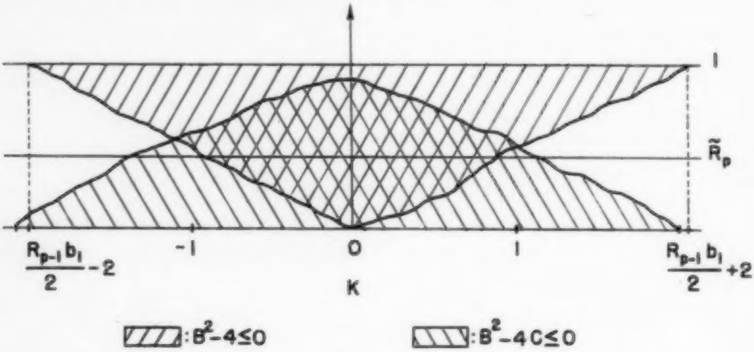

Figure 1

The lemmas to be proved are:

Lemma 4.1. *The set of points (r, K) for which $B^2 - 4 \leqslant 0$ and $0 \leqslant r \leqslant 1$, is convex in the direction of the K-axis and in the direction of the r-axis.*

Lemma 4.2. *The set of points (r, K) for which $B^2 - 4C \leqslant 0$ and $0 \leqslant r \leqslant 1$ is convex in the direction of the K-axis.*

Lemma 4.3. *The set of points (r, K) for which $B^2 - 4C \leqslant 0$, $B^2 - 4 \leqslant 0$, and $0 \leqslant r \leqslant 1$ is convex in the direction of the r-axis.*

Lemma 4.4. *If for any fixed $a_2, a_3, \ldots, a_p$, there exists a K and an $r = \alpha$, $0 < \alpha < 1$, for which both $B^2 - 4C = 0$ and $B^2 - 4 = 0$, then $\alpha \geqq \tilde{R}_p$.*

It should be noted that the continuity of the boundaries of the regions in Figure 1 follows directly from the continuity of the functions $B^2 - 4C$ and $B^2 - 4$ in the two variables r and K, where $0 < r \leqslant 1$, $0 < R_{p-1} < 1$.

Proof of Lemma 4.1. In the proof of this lemma we assume that $b_1 > 0$. The lemma remains valid for $b_1 < 0$ with obvious modifications in the argument.

(a) From (2.11) and (2.13) we note that if $K = 0$ then

$$B = \frac{-r\cos\phi(1 - R_{p-1}^2)}{(1 - r^2 R_{p-1})^2}$$

and therefore $B^2 - 4 < 0$ for all $0 < r < 1$.

(b) For fixed $a_2, \ldots, a_p: \nu, r$ we have

$$\frac{dB}{dK} = \frac{1 + r^2}{r}.$$

From (a) and (b) the convexity in the direction of the K-axis is immediate.

(c) If $r = 1$ then $B = 2$ if

$$K = 2 + \frac{R_{p-1}b_1}{2}$$

and $B = -2$ if

$$K = -2 + \frac{R_{p-1}b_1}{2}.$$

Thus from (b) we have $B^2 - 4 < 0$ for all

$$-2 + \frac{R_{p-1}b_1}{2} < K < 2 + \frac{R_{p-1}b_1}{2}, \qquad r = 1.$$

(d) For $K > 0$,

$$\lim_{r \to 0} B = +\infty.$$

(e) For fixed $a_2, \ldots, a_p: \nu, K$

$$\frac{dB}{dr} = -\frac{[K(1 - r^2)(1 - r^2R_{p-1}^2)^2 + b_1R_{p-1}r^2(1 - R_{p-1}^2)(1 + r^2R_{p-1}^2)]}{2r^2(1 - r^2R_{p-1}^2)^2}$$
$$= \frac{N(r)}{Q(r)}.$$

From (c), (d), and (e) we obtain the convexity in the direction of the r-axis of the set of points (r, K) for which $K > 0$, $B^2 - 4 < 0$, $0 < r < 1$.

(f) $dB/dr = 0$ implies that

$$(4.4)\quad N(r) \equiv KR_{p-1}^4r^6 - (KR_{p-1}^4 + 2KR_{p-1}^2 + b_1R_{p-1}^3 - b_1R_{p-1}^5)r^4 + (2KR_{p-1}^2 - b_1R_{p-1} + b_1R_{p-1}^3 + K)r^2 - K = 0.$$

From (3.4) we have $N(0) = K$, $N(1) = b_1R_{p-1}(1 - R^4{}_{p-1})$, and the product of the roots of $N(r)$ is

$$\frac{1}{R_{p-1}^4} > 1.$$

Thus, if $K < 0$ we see that $N(r)$ has but one root in the interval $0 < r < 1$.

(g) When $K < 0$ we also have the following relations: $B < 0$; $dB/dr < 0$ for r sufficiently small, $dB/dr < 0$ for $r = 1$, and $\lim_{r \to 0} B = -\infty$.

From (c), (f), and (g), we obtain the convexity in the direction of the r-axis of the sets of points (r, K) satisfying $K < 0$, $B^2 - 4C < 0$, $0 < r < 1$.

Proof of Lemma 4.2. (a) First we note that $\lim_{r\to 0} B^2 - 4C < 0$ if and only if $|K| < 2$, while for $r = 1$, $B^2 - 4C > 0$ for all K.

(b) For $|K| = 2$, $B^2 - 4C > 0$ for all $0 < r < 1$.

(c) Then, since $B^2 - 4C$ is a quadratic in K it follows that for fixed $a_2, \ldots, a_p, \nu, r$ there exist at most two values of K for which $B^2 - 4C = 0$.

Proof of Lemma 4.3. This is immediate since if for a particular choice of $a_2, \ldots, a_p, \nu, \phi, r$, we have $B^2 - 4C < 0$ then the P under consideration is a member of $T_1(r)$ and from paragraph (4.2) we see that $B^2 - 4C$ cannot be greater than zero for any smaller r unless we have $B^2 - 4 > 0$.

Proof of Lemma 4.4. For fixed $a_2, \ldots, a_p: \nu, \phi$ let $P = u(r, \theta) + iv(r, \theta)$. Then, from the definition of P we have $v(r, 0) = 0$ and $v(r, \pi) = 0$ for all $0 < r < 1$. Thus, $v_r(r, 0)$ and $v_r(r, \pi) = 0$ for all $0 < r < 1$. Now if we rewrite (1.9) as $Q(a_2, \ldots, a_p: r, \nu, \phi, \theta)\ (\cos^2\theta + B\cos\theta + C) = Q(\cos^2\theta + B\cos\theta + C)$, we have, since $C > 0$, $Q(a_2, \ldots, a_p, r, \nu, \phi, 0) = 0$ and $Q(a_2, \ldots, a_p: r, \nu, \phi, \pi) = 0$.

Any solution of the system $\{B^2 - 4C = 0,\ B^2 - 4 = 0\}$ is also a solution of the equivalent system $\{B^2 - 4 = 0,\ C = 1\}$. Let $r = \alpha$, $0 < \alpha < 1$ be a solution of this system for some particular $a_2, \ldots, a_p: \nu, \phi$. We have

$$\begin{aligned} v_\theta(r, \theta) &= (Q)(-B\sin\theta - 2\sin\theta\cos\theta) + (Q_\theta)(\cos^2\theta + B\cos\theta + C) \\ v_r(r, \theta) &= (Q)(B_r\cos\theta + C_r) \qquad\qquad\qquad + (Q_r)(\cos^2\theta + B\cos\theta + C) \end{aligned}$$

and, therefore, we have

$$\begin{aligned} v_\theta(\alpha, 0) &= (Q_\theta)(1 + B + C)]_{\theta=0} & v_r(\alpha, 0) &= 0, \\ v_\theta(\alpha, \pi) &= (Q_\theta)(1 - B - C)]_{\theta=\pi} & v_r(\alpha, \pi) &= 0. \end{aligned}$$

Thus for $r = \alpha$ either $v_\theta(\alpha, 0) = 0$ or $v_\theta(\alpha, \pi) = 0$, since B and C are independent of θ. This, however, implies that either $P'(\alpha) = 0$ or $P'(-\alpha) = 0$ for this choice of $a_2, \ldots, a_p: \nu, \phi$. Thus $\alpha \geqq \tilde{R}_p$ follows from (3.2).

Proof of the Main Theorem. From Lemmas 4.1 through 4.4 it is clear that for any choice of $a_2, \ldots, a_p$ no function P can belong to $T_1(r)$ if $r < \tilde{R}_p$. The proof then follows directly from the first two paragraphs of § 4, and formula (2.1).

5. The Class $T_2(1)$. Because of the discontinuity of the functions (2.11) and (2.12) at $r = 1$, $R_{p-1} = 1$ the derivation of the R_p, $p > 2$ employed in § 4 is not valid for the case $p = 2$ in which $R_{p-1} \equiv R_1 \equiv 1$. We present, therefore, in this section a rather simple proof of the validity for $p = 2$ of the Main Theorem. This proof is a modification of the proof found in the author's paper **(1)**.

When $p = 2$ we must have $b_1 > 0$ if statement (2.2) is to be compatible with Rogosinski's definition of the class $T_1(1)$, **(6)**.

(5.1) $$B = \frac{K(1+r^2)}{2r},$$

(5.2) $$C = \left(\frac{1-r^2}{2r}\right) - \frac{K(\cos\phi + \cos\nu)}{2} - \cos\phi\cos\nu,$$

and

(5.3) $$K = b_1 - 2\cos\phi = a_2 - 2\cos\nu - 2\cos\phi.$$

Equation (3.1) takes the form

(5.4) $$\left(\frac{\omega^2+1}{2\omega} + \frac{K}{2}\right)^2 - \left(\cos\phi\cos\nu + \frac{Ka_2}{4}\right) = 0, \qquad 0 < |\omega| < 1.$$

Solving (5.4) for ω, we obtain

(5.5) $$\omega_1 = a + (a^2-1)^{\frac{1}{2}},\ \omega_2 = a - (a^2-1)^{\frac{1}{2}},\ \omega_3 = b + (b^2-1)^{\frac{1}{2}},\ \omega_4 = b - (b^2-1)^{\frac{1}{2}},$$

where

(5.6) $$a = \frac{-K}{2} + (\cos\phi\cos\nu + \tfrac{1}{4}Ka_2)^{\frac{1}{2}}$$

and $$b = \frac{-K}{2} - (\cos\phi\cos\nu + \tfrac{1}{4}Ka_2)^{\frac{1}{2}}.$$

From (5.5) and (5.6) it is evident that to obtain $\bar{R}_2(a_2)$ we need only minimize the expression $|a| - (a^2-1)^{\frac{1}{2}}$, $|a| \geqq 1$, since $|a|$ and $|b|$ have the same maximum value.

The minimum of $|a| - (a^2-1)^{\frac{1}{2}}$ occurs when $|a|$ is maximum, that is, when $\phi = \nu = \pi$, $a_2 > 0$ or $\phi = \nu = 0$, $a_2 < 0$. Thus

(5.7) $$\bar{R}_2(a_2) = (|a_2| + 3) - ((|a_2| + 3)^2 - 1)^{\frac{1}{2}}.$$

We do not establish the validity of the Lemmas 4.1 to 4.4 for $p = 2$ since from (5.1) we have for fixed a_2, ν, and ϕ that

(5.8) $$\frac{d|B|}{dr} = \left|\frac{K}{2}\right|\left(\frac{r^2-1}{r^2}\right) < 0$$

for all $0 < r < 1$ and

(5.9) $$\frac{d(B^2-4C)}{dr} = \left(1 - \frac{K^2}{4}\right)\left(\frac{1-r^2}{4r^3}\right) > 0$$

for all $|K| < 2$, $0 < r < 1$.

From (5.7) we see that

$$\max_{|a_2|} \bar{R}_2(a_2) = 3 - 2\sqrt{2}.$$

If $r = 3 - 2\sqrt{2}$ and $|B| \leqq 2$ we have from (5.1) that $|K| \leqq 2/3$. Then from (5.8) we see that for $|K| > 2/3$ and $r < 3 - 2\sqrt{2}$ we have $|B| > 2$.

Next, from (5.1) and (5.2) we have for $r = 3 - 2\sqrt{2}$,

$$B^2 - 4C = 8(\tfrac{1}{4}K^2 - 1) + (\tfrac{1}{2}K + \cos\phi)(\tfrac{1}{2}K + \cos\nu)$$

which is readily seen to be negative if $|K| < 2/3$. Then from (5.9) we see that for $r < 3 - 2\sqrt{2}$ and $|K| < 2/3$ we have $B^2 - 4C < 0$.

Thus, as in § 4, it follows that for any fixed a_2, ν, $0 < \nu < \pi$, and all $r < \bar{R}_2(a_2)$ we have $P \in T_1(r)$. This establishes the Main Theorem for $p = 2$.

6. Sharpness. To show that our result is sharp over $\cup_p T_p(1)$ we give a function of class $T_2^*(1)$ which is typically real of order one for and only for $|z| < R_2(a_2) \equiv R(a_2)$ as defined in (5.7).

Consider the function

$$f(z) = \frac{z^3 + (a_2 + 4)z^2 + z}{(z+1)^4}; \qquad a_2 > 0, \; |z| < 1. \tag{6.1}$$

This function is a member of $T_2^*(1)$ and

$$f'(z) = \frac{(z-1)[z^2 + (2a_2 + 6)z + 1]}{(z+1)^5}. \tag{6.2}$$

From (5.2) it is readily seen that $f(z)$ cannot belong to $T_1(r)$ for any r greater than $(a_2 + 3) - ((a_2 + 3)^2 - 1)^{\frac{1}{2}} \equiv \bar{R}_2(a_2) \equiv R(a_2)$.

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CONTINUED FRACTIONS WITH ABSOLUTELY CONVERGENT EVEN OR ODD PART

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1. Introduction. The purpose of this paper is to give conditions under which the absolute convergence of the subsequence of odd or of even approximants to a continued fraction implies convergence of the continued fraction. In § 2 we consider the problem in general, and in § 3 we impose a condition which gives absolute convergence of the odd or of the even part of the continued fraction and state conditions which imply convergence of the continued fraction. If each of a and b is a complex number sequence, let $f(a)$ denote the continued fraction

$$\frac{1}{1} + \frac{a_1}{1} + \frac{a_2}{1} + \frac{a_3}{1} + \dots \tag{1.1}$$

and $g(b)$ denote the continued fraction

$$\frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3} + \dots . \tag{1.2}$$

Let $f(a)$ have the approximants f_p and $g(b)$ have the approximants g_p, where $f_p = A_p/B_p$ and $g_p = C_p/D_p$. Then

$$\begin{aligned} &A_0 = 0, A_1 = 1,\ A_{q+1} = A_q + a_q A_{q-1}, \\ &B_0 = 1, B_1 = 1,\ B_{q+1} = B_q + a_q B_{q-1}, \qquad q = 1, 2, 3, \dots, \end{aligned} \tag{1.3}$$

$$\begin{aligned} &C_0 = 0, C_1 = 1,\ C_{q+1} = b_{q+1} C_q + C_{q-1}, \\ &D_0 = 1, D_1 = b_1, D_{q+1} = b_{q+1} D_q + D_{q-1}, \quad q = 1, 2, 3, \dots . \end{aligned} \tag{1.4}$$

If $b_1 = 1$, $a_p \neq 0$, $1/b_{p+1} = a_p b_p$, $p = 1, 2, 3, \dots$, then (1.1) and (1.2) are equivalent in the sense that the two continued fractions have the same sequence of approximants. A well-known necessary condition for the convergence of $g(b)$ is that the series $\sum |b_p|$ diverge. Scott and Wall **(2)** investigated (1.1) by means of the systems of inequalities

$$\begin{aligned} &r_1|1 + a_1| \geqq (1 + r_{-1})|a_1| \\ &r_{2p+1}|1 + a_{2p} + a_{2p+1}| \geqq r_{2p+1} r_{2p-1}|a_{2p}| + |a_{2p+1}|, \quad p = 1, 2, 3, \dots, \end{aligned} \tag{1.5}$$

and

$$r_{2p}|1 + a_{2p-1} + a_{2p}| \geqq r_{2p} r_{2p-2}|a_{2p-1}| + |a_{2p}|, \quad p = 1, 2, 3, \dots, \tag{1.6}$$

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where r_p is a non-negative number, $p = -1, 0, 1, 2, \ldots$. They showed that if the r_p are subjected to certain restrictions (for example, $r_p = 1$, $p = -1, 0, 1, 2, \ldots$), then either some $a_p = 0$ and $f(a)$ converges, or else $a_p \neq 0$, $p = 1, 2, 3, \ldots$, and the divergence of the series $\sum|b_p|$ is necessary and sufficient for the convergence of $f(a)$. Lane and Wall **(1)** arrived at the same conclusion with the only restrictions on the r_p that r_{-1} and r_0 be distinct from zero by showing that if the even and odd parts of $f(a)$ converge absolutely, then $f(a)$ converges if, and only if, the series $\sum|b_p|$ diverges. This result is a consequence of two theorems **(1**, p. 371**)**, the first of which states that if the even and odd parts of $f(a)$ are convergent, the even (odd) part being absolutely convergent, and if the series $\sum|h_{2p-1}|$ (the series $\sum|h_{2p}|$) diverges, then $f(a)$ converges. Here

$$h_p = \frac{1}{1 + a_{p+1}h_{p+1}}, \qquad p = 1, 2, 3, \ldots.$$

The second of these theorems states that if there is a number M such that $|f_p| \leq M$, $p = 1, 2, 3, \ldots$, and no term of a is zero, then the two series $\sum|h_p|$ and $\sum|b_p|$ converge or diverge together. The question arises as to what restriction can be placed on the sequence b which would replace the condition that the series $\sum|h_{2p-1}|$ diverge in the first of these theorems. To answer this question by studying the relationships between the b_p and h_p appears difficult since the relationships are complicated:

$$b_1 = 1, \quad b_2 = -h_1, \quad b_3 = -h_2 \cdot \frac{1}{1 - h_1},$$

$$b_{2p+2} = -h_{2p+1} \cdot \frac{(1-h_1)(1-h_3)\ldots(1-h_{2p-1})}{(1-h_2)(1-h_4)\ldots(1-h_{2p})}, \qquad p = 1, 2, 3, \ldots.$$

$$b_{2p+3} = -h_{2p+2} \cdot \frac{(1-h_2)(1-h_4)\ldots(1-h_{2p})}{(1-h_1)(1-h_3)\ldots(1-h_{2p-1})},$$

We answer this question in § 2 by studying the continued fraction (1.2). The result is that if $\{g_{2p-1}\}$ ($\{g_{2p}\}$) converges absolutely and $\{g_{2p}\}$ ($\{g_{2p-1}\}$) converges, then $g(b)$ converges, provided the series $\sum|b_{2p-1}|$ (the series $\sum|b_{2p}|$) diverges. This result is stronger than expected in the light of results stated above. We use this result to construct a simple proof of a theorem of Van Vleck **(3)**: If there exists a positive number k such that $|\mathrm{Im}\, b_p| \leq k \cdot \mathrm{Re}\, b_p$, $p = 1, 2, 3, \ldots$, and $b_1 \neq 0$, then $g(b)$ converges if the series $\sum|b_p|$ diverges.

In § 3 we turn our attention to the systems of inequalities (1.5) and (1.6). The two main results obtained are:

(1) If (1.6) ((1.5)) holds, $0 < r_{2p-2} \leq 1$ ($0 < r_{2p-3} \leq 1$), $p = 1, 2, 3, \ldots$, and there exists a positive number M such that

$$M < \prod_{i=1}^{n} r_{2i-2} \left(M < \prod_{i=1}^{n} r_{2i-3}\right), \qquad n = 1, 2, 3, \ldots,$$

then $f(a)$ converges, provided a contains no zero term and the series $\sum |b_{2p-1}|$ (the series $\sum |b_{2p}|$) diverges;

(2) If (1.6) ((1.5)) holds and there exists a number r such that $0 < r_{2p-2} \leqq r < 1$ $(0 < r_{2p-3} \leqq r < 1)$, $p = 1, 2, 3, \ldots$, then $f(a)$ converges absolutely, provided a contains no zero term. Several examples are given in connection with these results.

2. Convergence of $g(b)$. Here we study conditions which give convergence of $g(b)$ when one of the sequences $\{g_{2p-1}\}$ and $\{g_{2p}\}$ converges absolutely. Throughout this section, theorems are stated and proofs are given for the case that $\{g_{2p-1}\}$ converges absolutely; the results for the other case follow by similar arguments.

LEMMA 2.1. *If z is a complex number sequence and there exists a non-negative integer N such that the series*

$$\sum \left| 1 - \frac{z_{N+p+1}}{z_{N+p}} \right|$$

converges, then z converges absolutely.

Proof. If $i > N$, then

$$\exp \left| 1 - \frac{z_{i+1}}{z_i} \right| > 1 + \left| \frac{z_{i+1}}{z_i} - 1 \right| > \left| \frac{z_{i+1}}{z_i} \right| - 1 + 1 = \left| \frac{z_{i+1}}{z_i} \right| .$$

Hence, if n is a positive integer, then

$$\begin{aligned} \left| \frac{z_{N+n+1}}{z_{N+1}} \right| &= \prod_{p=N+1}^{N+n} \left| \frac{z_{p+1}}{z_p} \right| < \prod_{p=N+1}^{N+n} \exp \left| 1 - \frac{z_{p+1}}{z_p} \right| \\ &= \exp \sum_{p=N+1}^{N+n} \left| 1 - \frac{z_{p+1}}{z_p} \right| < M, \end{aligned}$$

where

$$M = \exp \sum_{p=N+1}^{\infty} \left| 1 - \frac{z_{p+1}}{z_p} \right| ;$$

thus $|z_{N+n+1}| \leqq M |z_{N+1}|$. Therefore, there exists a number k such that $|z_p| < k$, $p = 1, 2, 3, \ldots$. Thus if q is an integer greater than $N + 1$, then

$$\sum_{p=N+1}^{q} |z_p - z_{p+1}| = \sum_{p=N+1}^{q} |z_p| \cdot \left| 1 - \frac{z_{p+1}}{z_p} \right| < k \cdot \sum_{p=N+1}^{q} \left| 1 - \frac{z_{p+1}}{z_p} \right| .$$

Therefore, z converges absolutely.

THEOREM 2.1. *If $\{g_{2p-1}\}$ converges absolutely to v and the series $\sum |b_{2p-1}|$ diverges, then there is an infinite subsequence of $\{g_{2p}\}$ which converges to v.*

Proof. Suppose $\{g_{2p-1}\}$ converges absolutely to v, the series $\sum |b_{2p-1}|$ diverges, and $\{g_{2p}\}$ contains no infinite subsequence which converges to v. There exists an integer S such that if $p > S$, then $D_{2p-1} \neq 0$. If $p > S$ and $D_{2p} \neq 0$, then

$$|g_{2p-1} - g_{2p}| = 1/|D_{2p-1}D_{2p}|.$$

Hence there exists a number k such that, if n is a positive integer, then $|D_{n+1}D_n| < k$, since $\{g_{2p}\}$ contains no infinite subsequence which converges to v. Suppose $p > S$. Then

$$|g_{2p+1} - g_{2p-1}| = \frac{|b_{2p+1}|}{|D_{2p+1}D_{2p-1}|}. \tag{2.1}$$

If there exists a number R such that $|D_{2p+1}D_{2p-1}| < R$, $p = 1, 2, 3, \ldots$, then if $n > S$ and m is a positive integer, it follows that

$$\sum_{p=n}^{n+m} |g_{2p+1} - g_{2p-1}| = \sum_{p=n}^{n+m} \frac{|b_{2p+1}|}{|D_{2p+1}D_{2p-1}|} > \frac{1}{R}\sum_{p=n}^{n+m} |b_{2p+1}|.$$

But this contradicts the fact that the series $\sum|b_{2p-1}|$ diverges. Thus if $R > 0$, there exists a positive integer i such that $|D_{2i+1}D_{2i-1}| > R$, and so either $|D_{2i+1}| > R^{\frac{1}{2}}$ or $|D_{2i-1}| > R^{\frac{1}{2}}$. Hence $\{D_{2p-1}\}$ contains an unbounded subsequence. From (2.1) it follows that if $p > S$ and $D_{2p} \neq 0$, then

$$|g_{2p+1} - g_{2p-1}| = \frac{|b_{2p+1}|\,|D_{2p}|^2}{|D_{2p+1}D_{2p}|\,|D_{2p}D_{2p-1}|}, \tag{2.2}$$

and so, by (1.4),

$$\left|1 - \frac{D_{2p+1}}{D_{2p-1}}\right| = \frac{|b_{2p+1}D_{2p}|}{|D_{2p-1}|} = \frac{|b_{2p+1}|\,|D_{2p}|^2}{|D_{2p}D_{2p-1}|},$$

and

$$|g_{2p+1} - g_{2p-1}| = \left|1 - \frac{D_{2p+1}}{D_{2p-1}}\right| \cdot \frac{1}{|D_{2p+1}D_{2p}|};$$

which means that

$$\left|1 - \frac{D_{2p+1}}{D_{2p-1}}\right| = |D_{2p+1}D_{2p}|\,|g_{2p+1} - g_{2p-1}| < k|g_{2p+1} - g_{2p-1}|.$$

On the other hand, if $p > S$ and $D_{2p} = 0$, then by (1.4), $D_{2p+1} = D_{2p-1}$, and so

$$\left|1 - \frac{D_{2p+1}}{D_{2p-1}}\right| = 0.$$

Hence the series

$$\sum \left|1 - \frac{D_{2(S+p)+1}}{D_{2(S+p)-1}}\right|$$

converges. Therefore, by the lemma, the sequence $\{D_{2p-1}\}$ converges absolutely. But this contradicts a statement proved above that $\{D_{2p-1}\}$ contains an unbounded subsequence. Therefore, our assumption that $\{g_{2p}\}$ contains no infinite subsequence which converges to v is proved false, and the theorem is established.

THEOREM 2.2. *If $\{g_{2p-1}\}$ converges absolutely, $\{g_{2p}\}$ converges, and the series $\sum|b_{2p-1}|$ diverges, then $g(b)$ converges. There exists a sequence a such that the odd part of $f(a)$ converges absolutely, the even part converges, and the series $\sum|b_p|$ diverges, and $f(a)$ does not converge.*

Proof. The first part of the theorem follows immediately from Theorem 2.1. We now construct a sequence a as in the second part of the theorem. Let

$$f_{2p-1} = 1/2^{p-1}$$
$$f_{2p} = \begin{cases} \dfrac{p+2}{p+3} & \text{if } p \text{ is odd,} \\[2ex] \dfrac{p+3}{p+2} & \text{if } p \text{ is even,} \end{cases} \qquad p = 1, 2, 3, \ldots .$$

If p is a positive integer, let a_{p+2} be the number determined by

$$-a_{p+2}\,(f_p - f_{p+2})\,(f_{p+1} - f_{p+3}) = (f_p - f_{p+1})\,(f_{p+2} - f_{p+3}),$$

and let $a_1 = \frac{1}{3}$ and $a_2 = -\frac{2}{3}$. Then $\{f_p\}_{p=1}^{\infty}$ is the sequence of approximants of $f(a)$ **(5)**. Clearly $f(a)$ does not converge, but the odd part of $f(a)$ converges absolutely while the even part converges, but not absolutely. We note that $|a_2/a_1| = 2$. Let p be a positive integer. Then

$$\frac{a_{2p+2}(f_{2p} - f_{2p+2})(f_{2p+1} - f_{2p+3})}{a_{2p+1}(f_{2p-1} - f_{2p+1})(f_{2p} - f_{2p+2})} = \frac{(f_{2p} - f_{2p+1})(f_{2p+2} - f_{2p+3})}{(f_{2p-1} - f_{2p})(f_{2p+1} - f_{2p+2})},$$

and so

$$\begin{aligned} \left|\frac{a_{2p+2}}{a_{2p+1}}\right| &= \left|\frac{f_{2p-1} - f_{2p+1}}{f_{2p+1} - f_{2p+3}}\right| \left|\frac{f_{2p} - f_{2p+1}}{f_{2p-1} - f_{2p}}\right| \left|\frac{f_{2p+2} - f_{2p+3}}{f_{2p+1} - f_{2p+2}}\right| \\ &= 2\left|\frac{f_{2p} - f_{2p+1}}{f_{2p-1} - f_{2p}}\right| \left|\frac{f_{2p+2} - f_{2p+3}}{f_{2p+1} - f_{2p+2}}\right| \\ &> 2. \end{aligned}$$

We note that if p is a positive integer, then $|f_{2p} - f_{2p-1}| < 2$, and so

$$\begin{aligned} \left|\frac{1}{a_{2p+1}}\right| &= \left|\frac{(f_{2p-1} - f_{2p+1})(f_{2p} - f_{2p+2})}{(f_{2p-1} - f_{2p})(f_{2p+1} - f_{2p+2})}\right| \\ &> \tfrac{1}{4}|f_{2p-1} - f_{2p+1}|\,|f_{2p} - f_{2p+2}| \\ &= \frac{1}{2^{p+2}}\,|f_{2p} - f_{2p+2}|. \end{aligned}$$

Therefore, if n is a positive integer, then

$$\begin{aligned} |b_{2n+2}| = \frac{\left|\prod_{p=1}^{n} a_{2p}\right|}{\left|\prod_{p=1}^{n+1} a_{2p-1}\right|} &= \frac{1}{|a_{2n+1}|} \prod_{p=1}^{n} \left|\frac{a_{2p}}{a_{2p-1}}\right| \\ &> 2^n \cdot \frac{1}{2^{n+2}}\,|f_{2n} - f_{2n+2}| \\ &= \tfrac{1}{4}|f_{2n} - f_{2n+2}|. \end{aligned}$$

Thus the series $\sum|b_{2p}|$ diverges. This completes the proof of the theorem.

We now use Theorem 2.2. in constructing a simple proof of the theorem of Van Vleck mentioned in the Introduction.

THEOREM 2.3. (Van Vleck) *If there exists a positive number k such that*

$$|\mathrm{Im}\, b_p| \leqq k \cdot \mathrm{Re}\, b_p, \qquad p = 1, 2, 3, \ldots,$$

and $b_1 \neq 0$, then $g(b)$ converges provided the series $\sum |b_p|$ diverges.

Proof. Let

$$t_p(u) = \frac{1}{b_p + u}, \qquad p = 1, 2, 3, \ldots.$$

Then we see that if p is a positive integer and $\mathrm{Re}\, u \geqq 0$, then $\mathrm{Re}\, t_p(u) \geqq 0$. Let H denote the half-plane $z + \bar{z} \geqq 0$. If n is a positive integer, let $T_n(u) = t_1 t_2 \ldots t_n(u)$. We note that $T_n(H)$ is a circular disc and we denote its radius by R_n. We also note that $T_{p+1}(H)$ is a subset of $T_p(H)$, $p = 1, 2, 3, \ldots$. We find that

$$T_n(u) = \frac{C_{n-1}u + C_n}{D_{n-1}u + D_n},$$

and

$$R_n = \frac{\frac{1}{2}}{Re\, \bar{D}_n D_{n-1}} = \frac{\frac{1}{2}}{\sum_{p=1}^{n} Re\, b_p |D_{p-1}|^2}.$$

Thus $T_n(0) = C_n/D_n = g_n$ and $T_n(\infty) = C_{n-1}/D_{n-1} = g_{n-1}$. Since $|b_p| \leqq \mathrm{Re}\, b_p + |\mathrm{Im}\, b_p| \leqq \mathrm{Re}\, b_p + k\, \mathrm{Re}\, b_p = (1 + k)\, \mathrm{Re}\, b_p$, $p = 1, 2, 3, \ldots$, it follows that

$$\begin{aligned}(1 + k)\, \mathrm{Re}\, \bar{D}_n D_{n-1} &= \sum_{p=1}^{n} (1 + k)\, \mathrm{Re}\, b_p |D_{p-1}|^2 > \sum_{p=1}^{n} |b_p|\, |D_{p-1}|^2 \\ &> |\bar{D}_n D_{n-1}|, \qquad n = 1, 2, 3, \ldots.\end{aligned}$$

Case 1. If $\mathrm{Re}\, \bar{D}_n D_{n-1} \to \infty$ as $n \to \infty$, then $R_n \to 0$ as $n \to \infty$ and $g(b)$ converges.

Case 2. If there exists a number M such that $(1 + k)\, \mathrm{Re}\, \bar{D}_p D_{p-1} < M$, $p = 1, 2, 3, \ldots$, then if m is a positive integer,

$$\begin{aligned}\sum_{p=1}^{m} |g_{p+1} - g_{p-1}| &= \sum_{p=1}^{m} \frac{|b_{p+1}|}{|D_{p+1}D_{p-1}|} = \sum_{p=1}^{m} \frac{|b_{p+1}|\, |D_p|^2}{|D_{p+1}D_p|\, |D_p D_{p-1}|} \\ &< L^2 \sum_{p=1}^{m} |b_{p+1}|\, |D_p|^2 < L^2 M,\end{aligned}$$

where $L = 2R_1$. Hence the even and odd parts of $g(b)$ converge absolutely. Thus, by Theorem 2.2, $g(b)$ converges since either the series $\sum |b_{2p-1}|$ or the series $\sum |b_{2p}|$ diverges. Therefore the theorem is established.

Remark 2.1. It is interesting to note in the above that actually $R_n \to 0$ as $n \to \infty$. If c is a positive number, there exists a positive integer N, such that, if $n > N$, then

$$|g_n - g_{n-1}| = \frac{1}{|\bar{D}_n D_{n-1}|} < c;$$

and so

$$|\bar{D}_n D_{n-1}| > \frac{1}{c}.$$

Thus

$$Re\, \bar{D}_n D_{n-1} > \frac{1}{c(1+k)}.$$

Therefore, $R_n \to 0$ as $n \to \infty$.

Remark 2.2. A curious corollary to Lemma 2.1 is that convergence of the series $\sum |C_p D_{p+1}|^{-1}$ implies convergence of the series $\sum |D_p D_{p+1}|^{-1}$. This is evident since, if p is a positive integer,

$$\left|1 - \frac{g_{p+1}}{g_p}\right| = \left|\frac{g_p - g_{p+1}}{g_p}\right| = \frac{|C_p/D_p - C_{p+1}/D_{p+1}|}{|C_p/D_p|} = \frac{1}{|C_p D_{p+1}|}.$$

Clearly, the converse is not true in general. For example, let $g_p = 2^{-p}$, $p = 1, 2, 3, \ldots$.

3. Convergence of $f(a)$**.** Here we study conditions which give convergence of $f(a)$ when one of the systems of inequalities (1.5) and (1.6) holds. We shall state the theorems of this section in terms of the system (1.6) only. Similar results are obtained when (1.5) holds.

THEOREM 3.1. *If* (1.6) *holds,* $r_0 \neq 0$, *and* $a_{2p} \neq 0$, $p = 1, 2, 3, \ldots$, *then* $\{f_{2p-1}\}$ *converges absolutely to a point* v (*a known result,* (**2**, *p.* 155)), *and if the series* $\sum |a_1 a_3 \ldots a_{2p-1}|\, |a_2 a_4 \ldots a_{2p}|^{-1}$ *diverges, then there exists an infinite subsequence of the sequence of approximants of the even part of* $f(a)$ *which converges to* v.

Proof. From (1.6) and (1.3) we observe that

$$(3.1) \qquad r_{2p}|B_{2p+1}| > |a_{2p}|\, |B_{2p-1}| + \left(\prod_{i=0}^{p} r_{2i}\right) \left|\prod_{i=1}^{p} a_{2i-1}\right|, \quad p = 1, 2, 3, \ldots.$$

By (1.6), $r_{2p} \neq 0$, $p = 1, 2, 3, \ldots$, since $a_{2p} \neq 0$, $p = 1, 2, 3, \ldots$, and from (3.1) it follows that $B_{2p+1} \neq 0$, $p = 1, 2, 3, \ldots$, (we note that $B_1 = 1$). Hence by (3.1),

$$|f_{2p+1} - f_{2p-1}| = \frac{|a_1 a_3 \ldots a_{2p-1}|}{|B_{2p+1} B_{2p-1}|}$$
$$< \frac{|a_2 a_4 \ldots a_{2p-2}|}{r_0 r_2 \ldots r_{2p-2}|B_{2p-1}|} - \frac{|a_2 a_4 \ldots a_{2p}|}{r_0 r_2 \ldots r_{2p}|B_{2p+1}|}, \qquad p = 2, 3, 4, \ldots.$$

Therefore,

$$\sum_{p=1}^{\infty} |f_{2p+1} - f_{2p-1}| < \frac{|a_1|}{|1 + a_1 + a_2|} + \frac{|a_2|}{r_0 r_2 |B_3|},$$

and $\{f_{2p-1}\}$ converges absolutely to a point v. We now require that the series

$$\Sigma|a_1a_3\ldots a_{2p-1}|\,|a_2a_4\ldots a_{2p}|^{-1}$$

diverge. Thus $a_p \neq 0$, $p = 1, 2, 3, \ldots$. Hence we can consider the continued fraction (1.2) equivalent to $f(a)$. Since $|b_{2p+1}| = |a_1a_3\ldots a_{2p-1}|\,|a_2a_4\ldots a_{2p}|^{-1}$ then by Theorem 2.1, the sequence of approximants of the even part of $f(a)$ contains an infinite subsequence convergent to v.

THEOREM 3.2. *If* (1.6) *holds and there exists a positive number M such that*

$$(1)\qquad 0 < r_{2p-2} \leq 1, \qquad p = 1, 2, 3, \ldots,$$

$$(2)\qquad M \leq \prod_{p=1}^{n} r_{2p-2}, \qquad n = 1, 2, 3, \ldots,$$

then $f(a)$ converges if either of the following conditions holds:

(i) $a_p \neq 0$, $p = 1, 2, 3, \ldots$, *and the series* $\sum|b_{2p-1}|$ *diverges,*
(ii) *some* $a_{2p} = 0$ *and no* $a_{2p-1} = 0$.

There exist a number sequence $\{r_{2p}\}_{p=0}^{\infty}$ and a sequence a which satisfy (1.6), (1), *and* (i) *such that $f(a)$ does not converge.*

Proof. Suppose $a_p \neq 0$, $p = 1, 2, 3, \ldots$. We consider the continued fraction (1.2) which is equivalent to $f(a)$, and note that

$$(3.2)\qquad \begin{aligned} B_{2p} &= a_1a_3\ldots a_{2p-1}D_{2p}, \\ B_{2p+1} &= a_2a_4\ldots a_{2p}D_{2p+1}, \end{aligned} \qquad p = 1, 2, 3, \ldots$$

From (3.1), (3.2), and (2), it follows that

$$r_{2p}|D_{2p+1}| \geq |D_{2p-1}| + M|b_{2p+1}|, \qquad p = 1, 2, 3, \ldots,$$

Hence

$$|D_{2p+1}| \geq M \sum_{i=1}^{p+1} |b_{2i-1}|,$$

since $D_1 = 1$ and $M \leq 1$. If n is a positive integer, then

$$\begin{aligned} |g_{2n+1} - g_{2n}| &= \frac{1}{|D_{2n}D_{2n+1}|} = \frac{|b_{2n+1}|}{|D_{2n+1}|\,|D_{2n+1} - D_{2n-1}|} \\ &\leq \frac{|b_{2n+1}|}{|D_{2n+1}|[|D_{2n+1}| - |D_{2n-1}|]} \leq \frac{1}{M|D_{2n+1}|} \\ &\leq \frac{1}{M^2\sum_{p=1}^{n+1}|b_{2p-1}|}. \end{aligned}$$

Therefore, since the odd part of $f(a)$ converges by Theorem 3.1, we see that $f(a)$ converges.

Suppose some $a_{2q} = 0$ and $a_{2p-1} \neq 0$, $p = 1, 2, 3, \ldots$. By (1.3) and (3.1) we see that $B_p \neq 0$, $p = 0, 1, 2, 3, \ldots$. Hence by a known theorem (**4**, p. 26), $f(a)$ converges. This completes the proof that $f(a)$ converges if either of the conditions (i) or (ii) holds.

Let

$$r_{2p-2} = \begin{cases} 1 \text{ if } p \text{ is odd} \\ \frac{1}{2} \text{ if } p \text{ is even,} \end{cases}$$

$$a_{2p-1} = a_{2p} = \begin{cases} 1 \text{ if } p \text{ is odd} \\ -2 \text{ if } p \text{ is even,} \end{cases} \qquad p = 1, 2, 3, \ldots .$$

Then (1), (1.6), and (i) are satisfied and $f(a)$ does not converge since the sequence of approximants of the even part of $f(a)$ contains an infinite subsequence of zero terms while the odd part of $f(a)$ does not converge to zero. This completes the proof of the theorem.

Remark 3.1. In (ii) of Theorem 3.2, the condition that $a_{2p-1} \neq 0$, $p = 1, 2, 3, \ldots$, in case some $a_{2p} = 0$ cannot be removed. We see this in the following example: let $r_{2p} = 1$, $p = 0, 1, 2, \ldots$, $a_1 = a_2 = 1$, $a_3 = a_6 = 0$, $a_4 = a_5 = -\frac{1}{2}$, $a_p = 1$, $p = 7, 8, 9, \ldots$. Clearly (1), (2), and (1.6) are satisfied. Simple calculations show that $B_6 = B_7 = 0$, and so by (1.3), $B_p = 0$, $p = 6, 7, 8, \ldots$.

Remark 3.2. In Theorem 3.2, in case $a_{2p} \neq 0$, $p = 1, 2, 3, \ldots$, but some $a_{2p-1} = 0$, the continued fraction does not necessarily converge, as shown by the following example: let $r_{2p} = 1$, $p = 0, 1, 2, 3, \ldots$, $a_1 = 0$, and

$$a_{2p} = a_{2p+1} = \begin{cases} -\frac{1}{2} \text{ if } p \text{ is odd} \\ 1 \text{ if } p \text{ is even,} \end{cases} \qquad p = 1, 2, 3, \ldots .$$

Then $B_{4p} = 0$, $p = 1, 2, 3, \ldots$.

THEOREM 3.3. *If* (1.6) *holds and there exists a number* r *such that* $0 < r_{2p-2} \leq r < 1$, $p = 1, 2, 3, \ldots$, *then* $f(a)$ *converges absolutely, provided one of the sequences* $\{a_{2p-1}\}$ *and* $\{a_{2p}\}$ *contains no zero term.*

Proof. Suppose $a_p \neq 0$, $p = 1, 2, 3, \ldots$. Again we consider the continued fraction (1.2) which is equivalent to $f(a)$. If n is a positive integer, then

$$\begin{aligned} |g_{2n+1} - g_{2n}| = \frac{1}{|D_{2n+1}D_{2n}|} &= \frac{|b_{2n+1}|}{|D_{2n+1}|\,|D_{2n+1} - D_{2n-1}|} \\ &= \frac{|b_{2n+1}|}{|D_{2n+1}D_{2n-1}|} \cdot \frac{1}{|D_{2n+1}/D_{2n-1} - 1|} \\ &\leq |g_{2n+1} - g_{2n-1}| \cdot \frac{1}{1/r_{2n} - 1} \\ &\leq |g_{2n+1} - g_{2n-1}| \cdot \frac{r}{1-r} . \end{aligned}$$

Thus it follows that $f(a)$ converges absolutely, since the odd part of $f(a)$ converges absolutely, as shown in the proof of Theorem 3.1.

Suppose some $a_{2q} = 0$ and $a_{2p-1} \neq 0$, $p = 1, 2, 3, \ldots$. Then by (3.1) and (1.3)

$$|B_{2p}| = |B_{2p+1} - a_{2p}B_{2p-1}| \geq |B_{2p+1}| - |a_{2p}B_{2p-1}| > 0, \quad p = 1, 2, 3, \ldots$$

Hence $B_p \neq 0$, $p = 1, 2, 3, \ldots$. Therefore $f(a)$ converges absolutely (**4**, p. 26).

On the other hand, suppose some $a_{2q-1} = 0$ and $a_{2p} \neq 0$, $p = 1, 2, 3, \ldots$. Since $r_{2p} \leq r < 1$, $p = 1, 2, 3, \ldots$, we see by (3.1) that

$$|B_{2p-1}| > |a_{2p}|\,|B_{2p-1}|, \qquad p = 1, 2, 3, \ldots,$$

and so $B_{2p-1} \neq 0$, $p = 1, 2, 3, \ldots$, since $a_2 B_1 \neq 0$. Therefore, as before, $B_p \neq 0$, $p = 1, 2, 3, \ldots$, and $f(a)$ converges absolutely.

Remark 3.3. We give an example of a number sequence $\{r_{2p}\}_{p=0}^{\infty}$ and an unbounded sequence a which satisfy the hypothesis of Theorem 3.3 as follows: let $r_{2p-2} = \frac{1}{2}$, $p = 1, 2, 3, \ldots$, and $a_1 = a_2 = 1$, $a_{2p-1} = -(2^p)$, $a_{2p} = -(2^{p-1}) + 1$, $p = 2, 3, 4, \ldots$.

Remark 3.4. There exist a number sequence $\{r_{2p}\}_{p=0}^{\infty}$ and a sequence a such that all of the conditions of Theorem 3.3. are satisfied except that each of the sequences $\{a_{2p-1}\}$ and $\{a_{2p}\}$ contains a zero term and $f(a)$ does not converge. Let $r_2 = \frac{1}{3}$, $r_{2p} = \frac{1}{2}$, $p = 0, 2, 3, 4, \ldots$, and $a_1 = a_4 = 0$, $a_2 = -\frac{1}{4}$, $a_3 = -\frac{3}{4}$, $a_p = 1$, $p = 5, 6, 7, \ldots$. Since $B_4 = 1 + a_1 + a_2 + a_3(1 + a_1) = 0$ and $B_5 = B_4 + a_4 B_3 = 0$, by (1.3), $B_p = 0$, $p = 4, 5, 6, \ldots$.

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GENERATING FUNCTIONS FOR HERMITE FUNCTIONS

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1. Introduction. Hermite's function $H_n(x)$ is defined for all complex values of x and n by

$$H_n(x) = \frac{2^n\Gamma(\frac{1}{2})}{\Gamma\left(\frac{1-n}{2}\right)} F\left(-\frac{n}{2};\frac{1}{2};x^2\right) + \frac{2^n\Gamma(-\frac{1}{2})}{\Gamma\left(-\frac{n}{2}\right)} xF\left(\frac{1-n}{2};\frac{3}{2};x^2\right)$$

$$= 2^n \sum_{k=0}^{\infty} \frac{\binom{n}{k}\Gamma\left(\frac{1}{2}\right) x^k}{\Gamma\left(\frac{1-n+k}{2}\right)},$$

where $F(\alpha;\gamma;x)$ is Kummer's function with the customary indices omitted. It satisfies the differential equation

$$\frac{d^2v}{dx^2} - 2x\frac{dv}{dx} + 2nv = 0, \tag{1.1}$$

of which

$$h_n(x) = e^{x^2}H_{-n-1}(ix)$$

is a second solution. Every solution of (1.1) is an entire function. The only linearly independent polynomial solutions are the Hermite polynomials $H_n(x)$, $n = 0, 1, 2 \ldots$.

The partial differential operator

$$L = \frac{\partial^2}{\partial x^2} - 2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}$$

annuls $u = y^n v(x)$ if, and only if, $v(x)$ satisfies (1.1). It follows that if $u = u(x,y)$ is annulled by L and is expressible as a series of powers of y, the coefficient of y^n must be a solution of (1.1). It so happens that the equation $Lu = 0$ admits a 5-parameter group of continuous transformations. Following the methods described in a previous paper **(5)** we shall use this group to obtain solutions of $Lu = 0$ and thence generating functions for the Hermite functions.

The results may also be expressed in terms of Weber's function $D_n(x)$ by means of the relation

$$H_n(x) = 2^{\frac{1}{2}n} e^{\frac{1}{2}x^2} D_n(\sqrt{2}\,x).$$

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2. Group of operators. The operators

$$(2.1)\quad \begin{aligned} &A = y\frac{\partial}{\partial y},\ B = y^{-1}\frac{\partial}{\partial x},\ C = y\left(-\frac{\partial}{\partial x} + 2x\right),\\ &B_2 = \frac{1}{2}y^{-2}\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right),\ C_2 = -\frac{1}{2}y^2\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 1 - 2x^2\right)\end{aligned}$$

satisfy the commutator relations

$$(2.2)\quad \begin{aligned} &[A, B] = -B, \quad [A, C] = C, \quad [C, B] = -2,\\ &[A, B_2] = -2B_2, \quad [A, C_2] = 2C_2, \quad [C_2, B_2] = -A - \tfrac{1}{2},\\ &[B, B_2] = 0, \quad [C, C_2] = 0, \quad [B, C_2] = C, \quad [B_2, C] = B\end{aligned}$$

and therefore generate, with the identity operator, a continuous group Γ.

A generates the trivial group $x' = x$, $y' = ty$ $(t \neq 0)$, which is used for purposes of normalization. The extended forms of the transformation groups generated by the other operators are described by

$$\begin{aligned} e^{bB}f(x, y) &= f(x + by^{-1}, y)\\ e^{\beta B_2}f(x, y) &= f\left(\frac{xy}{\sqrt{[y^2 - \beta]}}, \sqrt{[y^2 - \beta]}\right)\\ e^{cC}f(x, y) &= e^{2cxy - c^2y^2}f(x - cy, y)\\ e^{\gamma C_2}f(x, y) &= (1 + \gamma y^2)^{-\frac{1}{2}}\exp\left\{\frac{\gamma x^2y^2}{1 + \gamma y^2}\right\}f\left(\frac{x}{\sqrt{[1 + \gamma y^2]}}, \frac{y}{\sqrt{[1 + \gamma y^2]}}\right),\end{aligned}$$

where b, β, c, γ are arbitrary constants and $f(x, y)$ an arbitrary function. Hence

$$(2.3)\quad \begin{aligned} &e^{cC+\gamma C_2}e^{bB+\beta B_2}f(x, y)\\ &\qquad = (1 + \gamma y^2)^{-\frac{1}{2}}\exp\left\{\frac{2cxy - c^2y^2 + \gamma x^2y^2}{1 + \gamma y^2}\right\}f(\xi, \eta),\\ &\xi = \frac{b + xy + (b\gamma - c)y^2}{\{(1 + \gamma y^2)[(1 - \beta\gamma)y^2 - \beta]\}^{\frac{1}{2}}},\ \eta = \left\{\frac{(1 - \beta\gamma)y^2 - \beta}{1 + \gamma y^2}\right\}^{\frac{1}{2}}.\end{aligned}$$

The relation of the group Γ to the operator L of § 1 is indicated by the operator identities

$$(2.4)\quad \begin{aligned} -L &= CB - 2A, & -x^2L &= 4C_2B_2 - A^2 + A,\\ 4B_2 &= B^2 - y^{-2}L, & 4C_2 &= C^2 - y^2L,\end{aligned}$$

from which it follows that L is commutative with A, B, C and x^2L is commutative with A, B_2, C_2. Therefore every operator of the group Γ converts each solution of $Lu = 0$ into a solution. In particular we note that

$$(2.5)\quad \begin{aligned} A\{H_n(x)y^n\} &= nH_n(x)y^n, & A\{h_n(x)y^n\} &= nh_n(x)y^n;\\ B\{H_n(x)y^n\} &= 2nH_{n-1}(x)y^{n-1}, & B\{h_n(x)y^n\} &= -ih_{n-1}(x)y^{n-1};\\ C\{H_n(x)y^n\} &= H_{n+1}(x)y^{n+1}, & C\{h_n(x)y^n\} &= 2i(n + 1)h_{n+1}(x)y^{n+1}.\end{aligned}$$

3. Conjugate operators of the first order. We shall examine the functions annulled by L and

$$R = r_1 A + r_2 B + r_3 C + r_4 B_2 + r_5 C_2 + r_6,$$

where the r's are arbitrary constants, of which the first five do not vanish simultaneously. To this end we separate the operators R into conjugate classes with respect to the group Γ. We find as in (**5**, p. 1035) that

$$\begin{aligned}
&e^{aA}Be^{-aA} = e^{-a}B,\ e^{aA}Ce^{-aA} = e^{a}C,\ e^{aA}B_2e^{-aA} = e^{-2a}B_2,\ e^{aA}C_2e^{-aA} = e^{2a}C_2;\\
&e^{bB}A\,e^{-bB} = A + bB,\ e^{bB}Ce^{-bB} = C + 2b,\ e^{bB}C_2e^{-bB} = bC + C_2 + b^2;\\
&e^{cC}Ae^{-cC} = A - cC,\ e^{cC}Be^{-cC} = B - 2c,\ e^{cC}B_2e^{-cC} = -cB + B_2 + c^2;\\
&e^{\beta B_2}Ae^{-\beta B_2} = A + 2\beta B_2,\ e^{\beta B_2}Ce^{-\beta B_2} = C + \beta B,\ e^{\beta B_2}C_2e^{-\beta B_2} = \beta A + \beta^2 B_2\\
&\qquad\qquad + C_2 + \tfrac{1}{2}\beta;\\
&e^{\gamma C_2}Ae^{-\gamma C_2} = A - 2\gamma C_2,\ e^{\gamma C_2}Be^{-\gamma C_2} = B - \gamma C,\ e^{\gamma C_2}B_2e^{-\gamma C_2} = -\gamma A + B_2\\
&\qquad\qquad + \gamma^2 C_2 - \tfrac{1}{2}\gamma.
\end{aligned}$$

It follows that $I = r_1^2 - r_4 r_5$ is an invariant of R with respect to Γ.

Setting $S = e^{cC+\gamma C_2}e^{bB+\beta B_2}$, we have

$$\begin{aligned}
SAS^{-1} &= (1 - 2\beta\gamma)A + (b - 2c\beta)B + (2c\beta\gamma - c - b\gamma)C + 2\beta B_2\\
&\qquad + 2\gamma(\beta\gamma - 1)C_2 + 2c^2\beta - 2bc - \beta\gamma,\\
SBS^{-1} &= B - \gamma C - 2c,\\
SCS^{-1} &= \beta B + (1 - \beta\gamma)C + 2(b - c\beta),\\
SB_2S^{-1} &= -\gamma A - cB + c\gamma C + B_2 + \gamma^2 C_2 + c^2 - \tfrac{1}{2}\gamma,\\
SC_2S^{-1} &= \beta(1 - \beta\gamma)A + \beta(b - c\beta)B + (1 - \beta\gamma)(b - c\beta)C + \beta^2 B_2\\
&\qquad + (1 - \beta\gamma)^2 C_2 + (b - c\beta)^2 + \tfrac{1}{2}\beta(1 - \beta\gamma).
\end{aligned}$$

From these formulae it follows that for suitable choices of the constants $a, b, c, \beta, \gamma, \lambda, \nu, p$, and q, R is a conjugate of

(a)	$\lambda A - \nu$	if $I \neq 0$;
(b)	$pC + qB_2$	if $I = 0,\ r_1 r_2 \neq r_3 r_4$;
(c)	$\lambda B_2 - \nu$	if $I = 0,\ r_1 r_2 = r_3 r_4,\ r_4 \neq 0$ or $r_5 \neq 0$;
(d)	$\lambda B - \nu$	if $I = 0,\ r_1 = r_4 = r_5 = 0,\ r_2 \neq 0$ or $r_3 \neq 0$.

The identities (2.4) show that the last two cases do not require special consideration.

4. Generating functions for functions annulled by conjugates of $A - \nu$. Since $u_1 = y^\nu H_\nu(x)$, $u_2 = y^\nu e^{x^2} H_{-\nu-1}(ix)$ are linearly independent solutions of $Lu = 0$, $(A - \nu)\,u = 0$, where ν is an arbitrary constant, it follows from (2.3) that

$$\begin{aligned}
G_1(x, y) &= (1 + \gamma y^2)^{-(\nu+1)/2}\{(1 - \beta\gamma)y^2 - \beta\}^{\frac{1}{2}\nu}\\
&\qquad .\exp\left\{\frac{2cxy - c^2y^2 + \gamma x^2y^2}{1 + \gamma y^2}\right\} H_\nu(\xi),\\
G_2(x, y) &= (1 + \gamma y^2)^{-(\nu+1)/2}\{(1 - \beta\gamma)y^2 - \beta)^{\frac{1}{2}\nu}\\
&\quad .\exp\left\{\frac{(1 - \beta\gamma)x^2y^2 + (\beta c^2 - 2bc + b^2\gamma)y^2 + 2(b - \beta c)xy + b^2}{(1 - \beta\gamma y)^2 - \beta}\right\} H_{-\nu-1}(i\xi)
\end{aligned}$$

are linearly independent solutions of $Lu = 0$, $\{S(A - \nu)S^{-1}\}u = 0$. It suffices to examine G_1.

Case 1. $\beta = \gamma = c = 0$. Setting $b = 1$, we obtain, after simplification

$$H_\nu(x + y) = \sum_{n=0}^{\infty} \binom{\nu}{n} H_{\nu-n}(x)(2y)^n, \tag{4.1}$$

a Taylor expansion which may be derived directly from $H_\nu'(x) = 2\nu H_{\nu-1}(x)$.

Case 2. $\beta = \gamma = b = 0$. Setting $c = 1$, we have

$$y^\nu e^{2xy-y^2} H_\nu(x - y) = \sum_{n=0}^{\infty} \{a_n H_{\nu+n}(x) + b_n h_{\nu+n}(x)\} y^{\nu+n}.$$

Since the left member is annulled by $S(A - \nu)S^{-1} = A - C - \nu$, we obtain the recurrence relations

$$na_n = a_{n-1}, \quad nb_n = 2i(n+1)b_{n-1} \qquad (n = 1, 2, \ldots)$$

with the aid of (2.5). Cancelling y^ν and setting $y = 0$, we have $a_0 = 1$, $b_0 = 0$, whence $a_n = 1/n!$, $b_n = 0$. Hence (**4**, p. 85)

$$e^{2xy-y^2} H_\nu(x - y) = \sum_{n=0}^{\infty} \frac{1}{n!} H_{\nu+n}(x) y^n. \tag{4.2}$$

Case 3. $\beta = \gamma = 0$, $c \neq 0$. Setting $c = 1$, $b = -w/2$, we obtain with the aid of (4.1) and (4.2)

$$\begin{aligned} e^{2xy-y^2} H_\nu\left(x - y - \frac{w}{2y}\right) &= \sum_{n=0}^{\infty} \frac{1}{n!} F(-\nu; n+1; w) H_{\nu+n}(x) y^n \\ &+ \sum_{n=1}^{\infty} (-1)^n \binom{\nu}{n} F(n - \nu; n+1; w) H_{\nu-n}(x) w^n y^{-n}, \qquad (y \neq 0). \end{aligned} \tag{4.3}$$

If ν is a non-negative integer, this result may be written

$$\frac{y^\nu}{\nu!} e^{2xy-y^2} H_\nu\left(x - y - \frac{w}{2y}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} L_\nu^{(n-\nu)}(w) H_n(x) y^n, \tag{4.4}$$

where $L_\nu^{(\alpha)}(w)$ is the generalized Laguerre polynomial of degree ν.

Case 4. $\beta \neq 0$. Setting $\beta = -1$, $b = w$, $c = z$, we obtain

$$\begin{aligned} (1 + \gamma y^2)^{-(\nu+1)/2} \{1 + (1+\gamma)y^2\}^{\frac{1}{2}\nu} \exp\left\{\frac{2xyz - y^2z^2 + \gamma x^2y^2}{1 + \gamma y^2}\right\} H_\nu(\xi) \\ = \sum_{n=0}^{\infty} g_n H_n(x) y^n, \qquad |y| < \operatorname{Min}(|\gamma|^{-\frac{1}{2}}, |1+\gamma|^{-\frac{1}{2}}), \end{aligned} \tag{4.5}$$

where

$$\xi = \frac{w + xy + (\gamma w - z)y^2}{\{(1 + \gamma y^2)[1 + (1+\gamma)y^2]\}^{\frac{1}{2}}}.$$

By inspection of the left member it is evident that the coefficient of y^n is a polynomial in x; hence the second solution does not occur. Replacing x by $1/x$, y by xy, and then setting $x = 0$, we obtain

$$(4.6) \qquad e^{2yz+\gamma y^2}H_\nu(w+y) = \sum_{n=0}^{\infty} g_n(2y)^n,$$

a simple generating function for g_n. The explicit form of g_n may be found with the aid of (4.1) and (4.2):

$$(4.7) \quad g_n = \sum_{k=0}^{n} \binom{\nu}{n-k} \frac{(-\gamma)^{k/2}}{2^k k!} H_k\left(\frac{z}{(-\gamma)^{\frac{1}{2}}}\right) H_{\nu-n+k}(w) \qquad (\gamma \neq 0)$$

$$g_n = \sum_{k=0}^{n} \binom{\nu}{n-k} \frac{1}{k!} H_{\nu-n+k}(w) z^k \qquad (\gamma = 0).$$

In particular, when $\gamma = z = 0$ (**1**, p. 890)

$$(4.8) \qquad (1+y^2)^{\frac{1}{2}\nu} H_\nu\left(\frac{w+xy}{\sqrt{[1+y^2]}}\right) = \sum_{n=0}^{\infty} \binom{\nu}{n} H_{\nu-n}(w) H_n(x) y^n \quad (|y| < 1).$$

When $\gamma = -1$ and $z = -w$, the value of g_n may be obtained by comparing (4.6) with (4.2). Thus

$$(4.9) \qquad (1-y^2)^{-(\nu+1)/2} \exp\left\{\frac{2wxy - (x^2+w^2)y^2}{1-y^2}\right\} H_\nu\left(\frac{w-xy}{\sqrt{[1-y^2]}}\right)$$
$$= \sum_{n=0}^{\infty} \frac{H_{\nu+n}(w) H_n(x) y^n}{2^n n!}, \qquad (|y| < 1),$$

which reduces to Mehler's formula (**3**, p. 173) when $\nu = 0$ and to Feldheim's formula (**2**, p. 233) when $x = w$ and ν is an even number.

Case 5. $\beta = 0$, $\gamma \neq 0$. Setting $\gamma = -1$, $b = z$, $c = w$, we obtain with the aid of (4.1) and (4.9)

$$(4.10) \quad (1-y^2)^{-\frac{1}{2}(\nu+1)} \exp\left\{\frac{2wxy - (x^2+w^2)y^2}{1-y^2}\right\}$$
$$\cdot H_\nu\left(\frac{x-wy}{\sqrt{[1-y^2]}} + \frac{z\sqrt{[1-y^2]}}{y}\right) = \sum_{n=-\infty}^{\infty} g_n H_{\nu+n}(x)(y/2)^n,$$

where

$$g_n = \sum_{k=0}^{\infty} \binom{\nu}{k} \frac{1}{\Gamma(k+n+1)} H_{k+n}(w) z^k \qquad (n = 0, \pm 1, \pm 2 \ldots).$$

Moreover g_n has the generating function

$$(1 + z/y)^\nu e^{2wy - y^2} = \sum_{n=-\infty}^{\infty} g_n y^n, \qquad (|y| > |z|).$$

5. Generating functions annulled by conjugates of $3C - B_2$. In accordance with the analysis of § 3 we examine next the functions annulled by L and $pC + qB_2$, $pq \neq 0$. Only the ratio p/q is essential, and it proves convenient to choose $p = 3$, $q = -1$.

The general solution of $(3C - B_2)\, u = 0$, or

$$(x + 6y^3) \frac{\partial u}{\partial x} - y \frac{\partial u}{\partial y} = 12xy^3 u$$

is

$$u = e^{-6y^2(x+y^2)}f(\zeta), \qquad \zeta = 2xy + 3y^4.$$

This function is annulled by L if

$$\frac{d^2f}{d\zeta^2} - 3\zeta f = 0.$$

Two linearly independent solutions are given by

$$f = {}_0F_1\left(-; \frac{2}{3}; \frac{1}{3}\zeta^3\right), \qquad f = \zeta\, {}_0F_1\left(-; \frac{4}{3}; \frac{1}{3}\zeta^3\right).$$

Therefore, omitting the indices,

$$u_1 = e^{-6y^2(x+y^2)}F\left(-; \frac{2}{3}; \frac{1}{3}\zeta^3\right),$$

$$u_2 = e^{-6y^2(x+y^2)}\zeta F\left(-; \frac{4}{3}; \frac{1}{3}\zeta^3\right)$$

are linearly independent solutions of $Lu = 0$, $(3C - B_2)u = 0$. Their expansions in powers of y are readily obtained. On replacing y by $y^{1/2}$, we obtain

$$\text{(5.1)} \quad e^{-6y(x+y)}F\left(-; \frac{2}{3}; \frac{y}{3}(2x + 3y)^3\right) = \sum_{n=0}^{\infty} \frac{\Gamma(2/3)H_{3n}(x)}{n!\Gamma(n + 2/3)}\left(\frac{y}{3}\right)^n,$$

$$\text{(5.2)} \quad e^{-6y(x+y)}(2x + 3y)F\left(-; \frac{4}{3}; \frac{y}{3}(2x + 3y)^3\right) = \sum_{n=0}^{\infty} \frac{\Gamma(4/3)H_{3n+1}(x)}{n!\Gamma(n + 4/3)}\left(\frac{y}{3}\right)^n.$$

Applying S to u_1 and u_2, and setting $w = c + 3\beta$, $z = 2b + 3\beta^2$, we obtain the following functions annulled by L and $S(3C - B_2)S^{-1} = \gamma A + wB + (3 - \gamma w)C - B_2 - \gamma^2 C_2 + 3z - w^2 + \frac{1}{3}\gamma$:

$$\text{(5.3)} \quad (1 + \gamma y^2)^{-1/2} e^Y F\left(-; \frac{2}{3}; \frac{1}{3}X^3\right) = \sum_{n=0}^{\infty} a_n H_n(x) y^n \qquad (|y| < |\gamma|^{-1/2})$$

$$\text{(5.4)} \quad (1 + \gamma y^2)^{-1/2} e^Y X F\left(-; \frac{4}{3}; \frac{1}{3}X^3\right) = \sum_{n=0}^{\infty} b_n H_n(x) y^n \qquad (|y| < |\gamma|^{-1/2}),$$

where

$$X = z + \frac{2y(x - wy)}{1 + \gamma y^2} + \frac{3y^4}{(1 + \gamma y^2)^2},$$

$$Y = x^2 - \frac{3y^2 z + (x - wy)^2}{1 + \gamma y^2} - \frac{6y^3(x - wy)}{(1 + \gamma y^2)^2} - \frac{6y^6}{(1 + \gamma y^2)^3}.$$

Replacing x by $1/x$ and y by xy, and then setting $x = 0$, we obtain the following generating functions for a_n and b_n:

$$e^{2wy+\gamma y^2}F\left(-; \frac{2}{3}; \frac{1}{3}(2y + z)^3\right) = \sum_{n=0}^{\infty} a_n(2y)^n,$$

$$e^{2wy+\gamma y^2}(2y + z)F\left(-; \frac{4}{3}; \frac{1}{3}(2y + z)^3\right) = \sum_{n=0}^{\infty} b_n(2y)^n.$$

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GENERATING FUNCTIONS FOR BESSEL FUNCTIONS

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1. Introduction. On replacing the parameter n in Bessel's differential equation

$$x^2\frac{d^2v}{dx^2} + x\frac{dv}{dx} + (x^2 - n^2)v = 0 \tag{1.1}$$

by the operator $y(\partial/\partial y)$, the partial differential equation $Lu = 0$ is constructed, where

$$L = x^2\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial x} - y^2\frac{\partial^2}{\partial y^2} - y\frac{\partial}{\partial y} + x^2 = \left(x\frac{\partial}{\partial x}\right)^2 - \left(y\frac{\partial}{\partial y}\right)^2 + x^2. \tag{1.2}$$

This operator annuls $u(x, y) = v(x)y^n$ if, and only if, $v(x)$ satisfies (1.1) and hence is a cylindrical function of order n. Thus every generating function of a set of cylindrical functions is a solution of $Lu = 0$.

It is shown in § 2 that the partial differential equation $Lu = 0$ is invariant under a three-parameter Lie group. This group is then applied to the systematic determination of generating functions for Bessel functions, following the methods employed in two previous papers (**4**; **5**).

2. Group of operators. The operators

$$A = y\frac{\partial}{\partial y},\ B = y^{-1}\frac{\partial}{\partial x} + x^{-1}\frac{\partial}{\partial y},\ C = -y\frac{\partial}{\partial x} + x^{-1}y^2\frac{\partial}{\partial y}$$

satisfy the commutator relations $[A, B] = -B$, $[A, C] = C$, $[B, C] = 0$, and therefore generate a three-parameter Lie group. From these relations and the operator identity

$$-x^{-2}L = BC - 1, \tag{2.1}$$

where L is the operator (1.2), it follows that A, B, C are commutative with $x^{-2}L$ and therefore convert every solution of $Lu = 0$ into a solution. In particular

$$\begin{cases} AJ_n(x)y^n = nJ_n(x)y^n,\ AJ_{-n}(x)y^n = nJ_{-n}(x)y^n, \\ BJ_n(x)y^n = J_{n-1}(x)y^{n-1},\ BJ_{-n}(x)y^n = -J_{-n+1}(x)y^{n-1}, \\ CJ_n(x)y^n = J_{n+1}(x)y^{n+1},\ CJ_{-n}(x)y^n = -J_{-n-1}(x)y^{n+1}, \end{cases} \tag{2.2}$$

where n is an arbitrary complex number.

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The operator A generates the trivial group $x' = x$, $y' = ty$, $(t \neq 0)$, which is used for purposes of normalization. The extended form of the group generated by the commutative operators B, C is described by

$$(2.3)\quad e^{bB+cC}f(x, y) = f([(x - 2cy)(x + 2b/y)]^{\frac{1}{2}}, [y(xy + 2b)/(x - 2cy)]^{\frac{1}{2}}),$$

where b and c are arbitrary constants and $f(x, y)$ an arbitrary function, the signs of the radicals being chosen so that the right member reduces to $f(x, y)$ when $b = c = 0$. If $f(x, y)$ is annulled by L, so is the right member of (2.3).

3. Generating functions annulled by operators of the first order. Since $J_\nu(x)y^\nu$ is annulled by L and $A - \nu$, it follows from the operator identity

$$e^{bB+cC}Ae^{-bB-cC} = A + bB - cC$$

(**4**, p. 1035) and (2.3) that

$$\begin{aligned}(3.1)\quad G(x, y) &= e^{bB+cC}J_\nu(x)y^\nu \\ &= (xy + 2b)^{\frac{1}{2}\nu}(xy^{-1} - 2c)^{-\frac{1}{2}\nu}J_\nu([x - 2cy)(x + 2b/y)]^{\frac{1}{2}})\end{aligned}$$

is annulled by L and $A + bB + cC - \nu$. While any cylindrical function of order ν may be employed in place of $J_\nu(x)$, it is sufficient to confine attention to the Bessel functions of the first kind.

If $b = 0$, we choose $c = 1$, so that

$$G(x, y) = (xy)^\nu(x^2 - 2xy)^{-\frac{1}{2}\nu}J_\nu([x^2 - 2xy]^{\frac{1}{2}}) = \sum_{n=0}^{\infty} g_n J_{\nu+n}(x)y^{\nu+n}.$$

The indicated expansion is justified by the observation that $(xy)^{-\nu}G(x, y)$ is an entire function of x and y. Since G is annulled by $A - C - \nu$, we find, with the aid of (2.2), that $g_{n-1} = ng_n$ $(n = 1, 2, \dots)$. Multiplying G by $(xy)^{-\nu}$ and then setting $x = 0$, noting that

$$(3.2)\qquad x^{-\nu}J_\nu(x)]_{x=0} = \frac{1}{2^\nu\Gamma(\nu + 1)},$$

we have $g_0 = 1$; hence $g_n = 1/n!$. Thus

$$(3.3)\qquad x^\nu(x^2 - 2xy)^{-\frac{1}{2}\nu}J_\nu([x^2 - 2xy]^{\frac{1}{2}}) = \sum_{n=0}^{\infty} J_{\nu+n}(x)y^n/n!,$$

which may be identified with Lommel's first formula (**3**, p. 140).

If $c = 0$, we choose $b = 1$, whence

$$\begin{aligned}(3.4)\qquad G(x, y) &= (y^2 + 2y/x)^{\frac{1}{2}\nu}J_\nu([x^2 + 2x/y]^{\frac{1}{2}}) \\ &= (2 + xy)^\nu(x^2 + 2x/y)^{-\frac{1}{2}\nu}J_\nu([x^2 + 2x/y]^{\frac{1}{2}}).\end{aligned}$$

From the last expression it is evident that G has a Laurent expansion about $y = 0$:

$$G(x, y) = \sum_{n=-\infty}^{\infty} g_n J_n(x)y^n, \qquad |xy| < 2.$$

Since this function is annulled by $A + B - \nu$, we find, with the aid of (2.2), that $g_{n+1} = (\nu - n)g_n$, $(n = 0, \pm 1, \pm 2, \ldots)$. Setting $x = 0$, we have $g_0 = 1/\Gamma(\nu + 1)$; hence $g_n = 1/\Gamma(\nu - n + 1)$. Replacing y by y^{-1}, we obtain

$$(3.5)\quad (xy)^{-\nu}(x^2 + 2xy)^{\frac{1}{2}\nu}J_\nu([x^2 + 2xy]^{\frac{1}{2}}) = \sum_{n=-\infty}^{\infty} J_n(x)(-y)^n/\Gamma(\nu + n + 1), \qquad |2y| > |x|.$$

Writing (3.4) in the form

$$G(x, y) = (xy)^\nu(1 + 2/xy)^\nu(x^2 + 2x/y)^{-\frac{1}{2}\nu}J_\nu([x^2 + 2x/y]^{\frac{1}{2}}),$$

it is evident that $(xy)^{-\nu}G$ is expressible as a power series in y^{-1}, convergent for $|xy| > 2$. We obtain, after simplification,

$$(3.6)\qquad (1 + 2y/x)^{\frac{1}{2}\nu}J_\nu([x^2 + 2xy]^{\frac{1}{2}}) = \sum_{n=0}^{\infty} J_{\nu-n}(x)y^n/n!, \qquad |2y| < |x|,$$

which may be identified with Lommel's second formula (**3**, p. 140).

If $bc \neq 0$, it proves convenient to choose $b = \frac{1}{2}w$, $c = -\frac{1}{2}w$, whence

$$(3.7)\quad G(x, y) = (w + xy)^{\frac{1}{2}\nu}(w + x/y)^{-\frac{1}{2}\nu}J_\nu([w^2 + x^2 + wx(y + y^{-1})]^{\frac{1}{2}})$$
$$= \sum_{n=-\infty}^{\infty} g_nJ_n(x)y^n, \qquad |xy| < |w|.$$

Replacing y by $2y/x$ and then setting $x = 0$, we obtain, with the aid of (3.2),

$$(1 + 2y/w)^{\frac{1}{2}\nu}J_\nu([w^2 + 2wy]^{\frac{1}{2}}) = \sum_{n=0}^{\infty} g_ny^n/n!, \qquad |2y| < |w|.$$

Comparing with (3.6), we infer that $g_n = J_{\nu-n}(w)$, $(n = 0, 1, 2, \ldots)$. Similarly, replacing y by $x/2y$ and then setting $x = 0$, we obtain

$$w^\nu(w^2 + 2wy)^{\frac{1}{2}\nu}J_\nu([w^2 + 2wy]^{\frac{1}{2}}) = \sum_{n=0}^{\infty} g_{-n}(-y)^n/n!.$$

Comparing with (3.3) we conclude that $g_{-n} = J_{\nu+n}(w)$, $(n = 0, 1, 2, \ldots)$. Hence

$$(3.8)\quad (w + xy)^{\frac{1}{2}\nu}(w + x/y)^{-\frac{1}{2}\nu}J_\nu([w^2 + x^2 + wx(y + y^{-1})]^{\frac{1}{2}})$$
$$= \sum_{n=-\infty}^{\infty} J_{\nu-n}(w)J_n(x)y^n, \qquad |xy| < |w|,$$

which may be identified with Graf's addition theorem (**3**, p. 359) by substituting $y = -e^{-i\phi}$. Another expansion of (3.7), valid for $|xy| > |w|$, may be obtained from (3.8) by replacing y by y^{-1}, interchanging x and w, and multiplying by y^ν.

We have now obtained, in normalized form, functions annulled by L and differential operators of the first order of the form $r_1A + r_2B + r_3C + r_4$, where the r's are constants and $r_1 \neq 0$. Generating functions annulled by $r_2B + r_3C + r_4$ are not included in (3.1) but may be derived independently.

Since $[B, C] = 0$, we seek a solution of the simultaneous equations $(B - 1)u = 0$, $(C - 1)u = 0$. This solution is annulled by $r_2B + r_3C + r_4$, normalized so that $r_2 + r_3 + r_4 = 0$. By (2.1) it is also annulled by L. We find the solution to be the familiar generating function

$$e^{\frac{1}{2}x(y-y^{-1})} = \sum_{n=-\infty}^{\infty} J_n(x)y^n \tag{3.9}$$

of the Bessel functions of integral order.

4. Generating functions annulled by $A(B - C) + \frac{1}{2}(B + C) + 4\alpha - 1$. By a suitable choice of new variables the equation $Lu = 0$ may be transformed into one solvable by separation of variables. A solution so obtained, if possessed of suitable analytic properties, provides a generating function for Bessel functions. We shall present several examples.

Choosing new variables

$$\xi = \tfrac{1}{2}x(y^{-1} - y + 2i), \eta = \tfrac{1}{2}x(y^{-1} - y - 2i),$$

the equation $Lu = 0$ is transformed into

$$4\xi \frac{\partial^2 u}{\partial \xi^2} - 4\eta \frac{\partial^2 u}{\partial \eta^2} + 2\frac{\partial u}{\partial \xi} - 2\frac{\partial u}{\partial \eta} - (\xi - \eta)u = 0.$$

Four linearly independent solutions are obtained by separation of variables:

$$\begin{aligned}
u_1 &= e^{-\frac{1}{2}(\xi+\eta)}{}_1F_1(\alpha; \tfrac{1}{2}; \xi){}_1F_1(\alpha; \tfrac{1}{2}; \eta),\\
u_2 &= \xi^{\frac{1}{2}}e^{-\frac{1}{2}(\xi+\eta)}{}_1F_1(\alpha + \tfrac{1}{2}; 3/2; \xi){}_1F_1(\alpha; \tfrac{1}{2}; \eta),\\
u_3 &= \eta^{\frac{1}{2}}e^{-\frac{1}{2}(\xi+\eta)}{}_1F_1(\alpha; \tfrac{1}{2}; \xi){}_1F_1(\alpha + \tfrac{1}{2}; 3/2; \eta),\\
u_4 &= (\xi\eta)^{\frac{1}{2}}e^{-\frac{1}{2}(\xi+\eta)}{}_1F_1(\alpha + \tfrac{1}{2}; 3/2; \xi){}_1F_1(\alpha + \tfrac{1}{2}; 3/2; \eta),
\end{aligned}$$

where α is an arbitrary constant. These functions are also annulled by

$$\begin{aligned}
&4\xi \frac{\partial^2}{\partial \xi^2} + 2\frac{\partial}{\partial \xi} - \xi + 1 - 4\alpha\\
&\quad = 4\xi(\xi - \eta)^{-2}L - A(B - C) - \tfrac{1}{2}(B + C) + 1 - 4\alpha,
\end{aligned}$$

where A, B, C are the operators of § 2, and hence by

$$A(B - C) + \tfrac{1}{2}(B + C) + 4\alpha - 1.$$

This operator provides recurrence relations for the coefficients of the expansions of the generating functions; but these relations will not be used.

When expressed in terms of x and y, the function u_1 is seen to have a Laurent expansion about $y = 0$:

$$\begin{aligned}
&e^{\frac{1}{2}x(y-y^{-1})}{}_1F_1(\alpha; \tfrac{1}{2}; \tfrac{1}{2}x/y - \tfrac{1}{2}xy + ix){}_1F_1(\alpha; \tfrac{1}{2}; \tfrac{1}{2}x/y - \tfrac{1}{2}xy - ix)\\
&\quad = \sum_{n=-\infty}^{\infty} g_n J_n(x)y^n.
\end{aligned}$$

Replacing y by $2y/x$ and then setting $x = 0$, we have by (3.2)

$$e^y[{}_1F_1(\alpha;\tfrac{1}{2};-y)]^2=\sum_{n=0}^{\infty} g_n y^n/n!.$$

By Kummer's formula,

$$e^y[{}_1F_1(\alpha;\tfrac{1}{2};-y)]^2={}_1F_1(\alpha;\tfrac{1}{2};-y)\,{}_1F_1(\tfrac{1}{2}-\alpha;\tfrac{1}{2};y).$$

The expansion of the right member in powers of y may be obtained with the aid of Chaundy's formula

$${}_1F_1(a;c;-y)\,{}_1F_1(a';c';y)=\sum_{n=0}^{\infty}\frac{(a)_n(-y)^n}{n!(c)_n}\,{}_3F_2\left[\begin{matrix}a',1-c-n;1\\ c',1-a-n\end{matrix}\right]$$

(**2**, p. 70). However, this expansion may be expressed in a more suitable form by means of the transformation formula

$${}_3F_2\left[\begin{matrix}\alpha_1,\alpha_2,\alpha_3;1\\ \beta_1,\beta_2\end{matrix}\right]=\frac{\Gamma(\beta_2)\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3)}{\Gamma(\beta_2-\alpha_3)\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2)}\times{}_3F_2\left[\begin{matrix}\beta_1-\alpha_1,\beta_1-\alpha_2,\alpha_3;1\\ \beta_1,\beta_1+\beta_2-\alpha_1-\alpha_2\end{matrix}\right]$$

(**1**, p. 98), whence

$$\text{(4.1)}\quad {}_1F_1(a;c;-y)\,{}_1F_1(a';c';y)=\sum_{n=0}^{\infty}\frac{(c+c'-a-a')_n}{(c)_n}y^n\,{}_3F_2\left[\begin{matrix}c'-a',c+c'+n-1,-n;1\\ c',c+c'-a-a'\end{matrix}\right].$$

Thus

$$e^y[{}_1F_1(\alpha;\tfrac{1}{2};-y)]^2=\sum_{n=0}^{\infty}{}_3F_2(\alpha,n,-n;\tfrac{1}{2};\tfrac{1}{2};1)y^n/n!,$$

and g_n is determined for $n=0,1,2,\ldots$. Since the generating function is unaltered when y is replaced by $-y^{-1}$, $g_{-n}=g_n$. Hence

$$\text{(4.2)}\quad e^{\frac{1}{2}x(y-y^{-1})}{}_1F_1(\alpha;\tfrac{1}{2};\tfrac{1}{2}x/y-\tfrac{1}{2}xy+ix)\,{}_1F_1(\alpha;\tfrac{1}{2};\tfrac{1}{2}x/y-\tfrac{1}{2}xy-ix)=\sum_{n=-\infty}^{\infty}{}_3F_2(\alpha,n,-n;\tfrac{1}{2};\tfrac{1}{2};1)J_n(x)y^n.$$

Since $\xi^{\frac{1}{2}}=(\frac{1}{2}x)^{\frac{1}{2}}(y^{-\frac{1}{2}}+iy^{\frac{1}{2}})$, u_2 has an expansion of the form

$$\sum_{n=-\infty}^{\infty}[a_nJ_{n+\frac{1}{2}}(x)+b_nJ_{-n-\frac{1}{2}}(x)]y^{n+\frac{1}{2}}.$$

Applying the methods described above, we obtain, after multiplying by $(2y/x)^{\frac{1}{2}}$

$$\text{(4.3)}\quad (1+iy)e^{\frac{1}{2}x(y-y^{-1})}{}_1F_1(\alpha+\tfrac{1}{2};3/2;\tfrac{1}{2}x/y-\tfrac{1}{2}xy+ix)\times{}_1F_1(\alpha;\tfrac{1}{2};\tfrac{1}{2}x/y-\tfrac{1}{2}xy-ix)$$
$$=(\pi/2x)^{\frac{1}{2}}\sum_{n=0}^{\infty}{}_3F_2(\alpha,n+1,-n;1,\tfrac{1}{2};1)J_{n+\frac{1}{2}}(x)[iy^{n+1}+(-y)^{-n}].$$

Replacing i by $-i$, we obtain the expansion which arises similarly from u_3.

Since $(\xi\eta)^{\frac{1}{2}} = \frac{1}{2}x(y + y^{-1})$, u_4 has a Laurent expansion about $y = 0$. We obtain, after replacing α by $\alpha - \frac{1}{2}$,

$$\begin{aligned}(4.4)\quad & \tfrac{1}{2}x(y + y^{-1})e^{\frac{1}{2}x(y-y^{-1})}{}_1F_1(\alpha; 3/2; \tfrac{1}{2}x/y - \tfrac{1}{2}xy + ix) \\ & \times {}_1F_1(\alpha; 3/2; \tfrac{1}{2}x/y - \tfrac{1}{2}xy - ix) \\ & = \sum_{n=-\infty}^{\infty} n\, {}_3F_2(\alpha, n + 1, 1 - n; 3/2; 3/2; 1)J_n(x)y^n.\end{aligned}$$

With the aid of these results the elementary solutions of the three-dimensional wave equation in parabolic cylindrical co-ordinates may be expressed in terms of cylindrical wave functions.

5. Generating functions annulled by $B^2 + 8CA + 4C$. When we choose new variables $\xi = xy - (x/y)^{\frac{1}{2}}$, $\eta = xy + (x/y)^{\frac{1}{2}}$, the equation $Lu = 0$ becomes

$$\frac{\partial^2 u}{\partial\xi^2} - \frac{\partial^2 u}{\partial\eta^2} + \frac{1}{4}(\xi - \eta)u = 0.$$

The following solutions are obtained by separation of variables:

$$\begin{aligned}u_1 &= {}_0F_1(2/3; -[\xi + z]^3/36)\,{}_0F_1(2/3; -[\eta + z]^3/36), \\ u_2 &= (\xi + z)\,{}_0F_1(4/3; -[\xi + z]^3/36)\,{}_0F_1(2/3; -[\eta + z)^3/36), \\ u_3 &= (\eta + z)\,{}_0F_1(2/3; -[\xi + z]^3/36)\,{}_0F_1(4/3; -[\eta + z]^3/36), \\ u_4 &= (\xi + z)(\eta + z)\,{}_0F_1(4/3; -[\xi + z]^3/36)\,{}_0F_1(4/3; -[\eta + z]^3/36\end{aligned}$$

where z is an arbitrary constant. These functions are also annulled by

$$\frac{\partial^2}{\partial\xi^2} + \frac{1}{4}(\xi + z) = 2\xi[(\xi - \eta)(\xi^2 - \eta^2)]^{-1}L + \frac{B^2}{16} + \tfrac{1}{2}CA + \frac{C}{4} + \frac{z}{4}$$

and hence by $R = B^2 + 8CA + 4C + 4z$.

The functions u_1 and u_4 have expansions of the form

$$\sum_{n=-\infty}^{\infty} g_n J_n(x)y^n.$$

Applying R, we obtain the recurrence relation

$$g_{n+2} + 4zg_n + 4(2n - 1)g_{n-1} = 0 \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

by means of (2.2). No explicit solution is available for arbitrary z. A solution is readily obtained for $z = 0$. We find that

$$\begin{aligned}(5.1)\quad & {}_0F_1(2/3; -[xy - (x/y)^{\frac{1}{2}}]^3/36)\,{}_0F_1(2/3; -[xy + (x/y)^{\frac{1}{2}}]^3/36) \\ & = \sum_{m=-\infty}^{\infty} (-24)^m\Gamma(m + 1/6)J_{3m}(x)y^{3m}/\Gamma(1/6),\end{aligned}$$

$$\begin{aligned}(5.2)\quad & \frac{1}{8}(x^2y^2 - x/y)\,{}_0F_1(4/3; -[xy - (x/y)^{\frac{1}{2}}]^3/36) \\ & \times {}_0F_1(4/3; -[xy + (x/y)^{\frac{1}{2}}]^3/36) \\ & = \sum_{m=-\infty}^{\infty} (-24)^m\Gamma(m + 5/6)J_{3m+2}(x)y^{3m+2}/\Gamma\left(\frac{5}{6}\right)\end{aligned}$$

For u_2 we obtain similarly

$$(5.3)\quad [xy - (x/y)^{\frac12}]\,{}_0F_1(4/3; -[xy - (x/y)^{\frac12}]^3/36)\; {}_0F_1(2/3; -[xy + (x/y)^{\frac12}]^3/36)$$
$$= -(\pi/2)^{\frac12}\sum_{m=0}^{\infty}(-24)^{-m}J_{3m+1/2}(x)y^{-3m-1/2}/m!$$
$$+ 2\sum_{m=-\infty}^{\infty}(-24)^{m}\Gamma(m+\tfrac12)J_{3m+1}(x)y^{3m+1}/\Gamma(\tfrac12).$$

6. Generating functions annulled by $A^2 + \alpha(2CA + C) + \beta C^2 - \nu^2$. If we choose new variables

$$\xi = \tfrac12[(x^2+2a^2xy)^{\frac12} - (x^2+2b^2xy)^{\frac12}],$$
$$\eta = \tfrac12[(x^2+2a^2xy)^{\frac12} + (x^2+2b^2xy)^{\frac12}], \qquad (a^2 \neq b^2),$$

where a and b are constants and the signs of the radicals are chosen so that $\xi = 0$, $\eta = x$ when $y = 0$, the equation $Lu = 0$ becomes

$$\xi^2\frac{\partial^2 u}{\partial\xi^2} - \eta^2\frac{\partial^2 u}{\partial\eta^2} + \xi\frac{\partial u}{\partial\xi} - \eta\frac{\partial u}{\partial\eta} + (\xi^2 - \eta^2)u = 0.$$

Comparing with (1.1), it follows that L annuls the four functions $J_{\pm\nu}(\xi)J_{\pm\nu}(\eta)$, where ν is arbitrary. These functions are also annulled by

$$\xi^2\frac{\partial^2}{\partial\xi^2} + \xi\frac{\partial}{\partial\xi} + \xi^2 - \nu^2 = \xi^2(\xi^2 + 2c\xi\eta + \eta^2)^{-1}L + A^2 + \tfrac12(a^2+b^2)(2CA + C) + a^2b^2C^2 - \nu^2,$$

where $c = (a^2+b^2)/(a^2-b^2)$, and hence by

$$R = A^2 + \tfrac12(a^2+b^2)\,(2CA + C) + a^2b^2C^2 - \nu^2.$$

Employing the methods described previously, and applying the well-known formulae

$$J_\mu(\alpha z)J_\nu(\beta z) = \frac{(\frac12\alpha z)^\mu(\frac12\beta z)^\nu}{\Gamma(\nu+1)}\sum_{n=0}^{\infty}\frac{(-1)^n(\frac12\alpha z)^{2n}}{n!\Gamma(\mu+n+1)} \times F(-n, -\mu-n; \nu+1; \beta^2/\alpha^2),$$

$$F(\alpha,\beta;\gamma;z) = (1-z)^{-\alpha}F\left(\alpha, \gamma-\beta; \gamma; \frac{z}{z-1}\right),$$

the following results are obtained:

$$(6.1)\quad 2^{2\nu}\Gamma(\nu+1)(a^2-b^2)^{-\nu}J_\nu(\xi)J_\nu(\eta)$$
$$= \sum_{n=0}^{\infty}\frac{(ab)^n}{n!}F\left(-n, n+2\nu+1; \nu+1; \frac{(a+b)^2}{4ab}\right)J_{\nu+n}(x)y^{\nu+n},$$
$$(a^2 \neq b^2,\ ab \neq 0,\ \nu \neq -1, -2, \ldots).$$

(6.2) $$2^{2\nu}\Gamma(\nu+1)(a^2-b^2)^{-\nu}J_\nu(\xi)J_{-\nu}(\eta)$$
$$= \sum_{n=0}^{\infty} \frac{(-ab)^n}{n!} F\left(-n, n+2\nu+1; \nu+1; \frac{(a+b)^2}{4ab}\right) J_{-\nu-n}(x)y^{\nu+n},$$
$$|y| < \mathrm{Min}(|x/2a^2|, |x/2b^2|), \ (a^2 \neq b^2, ab \neq 0, \nu \neq -1, -2, \ldots).$$

(6.3) $$\Gamma(\nu+1)(a+b)^{\nu}(a-b)^{-\nu}J_\nu(\xi)J_{-\nu}(\eta)$$
$$= \sum_{n=-\infty}^{\infty} \frac{(-ab)^n}{\Gamma(n-\nu+1)} F\left(-n, n+1; \nu+1; -\frac{(a-b)^2}{4ab}\right) J_n(x)y^n,$$
$$|y| > \mathrm{Max}(|x/2a^2|, |x/2b^2|), \ (a^2 \neq b^2, ab \neq 0, \nu \neq -1, -2, \ldots),$$

and the left member has the value $(\sin \nu\pi)/\nu\pi$ when $x = 0$.

The excluded case $ab = 0$ may be treated similarly. Setting $a = 0$, $b^2 = -2$, the following generating functions, annulled by $A^2 - 2CA - C - \nu^2$, are obtained:

(6.4) $$J_\nu(\tfrac{1}{2}[x-(x^2-4xy)^{\frac{1}{2}}])J_\nu(\tfrac{1}{2}[x+(x^2-4xy)^{\frac{1}{2}}])$$
$$= \sum_{n=0}^{\infty} \frac{1}{\Gamma(\nu+n+1)} \binom{2\nu+2n}{n} J_{\nu+n}(x)(y/2)^{\nu+n},$$

(6.5) $$J_\nu(\tfrac{1}{2}[x-(x^2-4xy)^{\frac{1}{2}}])J_{-\nu}(\tfrac{1}{2}[x+(x^2-4xy)^{\frac{1}{2}}])$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(\nu+n+1)} \binom{2\nu+2n}{n} J_{-\nu-n}(x)(y/2)^{\nu+n}, \ |4y| < |x|.$$

(6.6) $$e^{\nu\pi i}J_\nu(\tfrac{1}{2}[x-(x^2-4xy)^{\frac{1}{2}}])J_{-\nu}(\tfrac{1}{2}[x+(x^2-4xy)^{\frac{1}{2}}])$$
$$= \sum_{n=-\infty}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(n+1-\nu)\Gamma(n+1+\nu)} J_n(x)(2y)^n$$
$$+ i\pi^{-1}\sin\nu\pi \sum_{n=0}^{\infty} \frac{(\frac{1}{2}-\nu)_n(\frac{1}{2}+\nu)_n}{n!} J_{n+\frac{1}{2}}(x)(2y)^{-n-\frac{1}{2}}, \ |4y| > |x|,$$

where the left member has the value $(\sin \nu\pi)/\nu\pi$ when $x = 0$. Formulae (6.4) and (6.5) are limiting cases of formulae (6.1) and (6.2) respectively.

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SERIES OF PRODUCTS OF BESSEL POLYNOMIALS

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1. Introduction. The Bessel polynomials, which arise as solution of the classical wave equation in spherical co-ordinates, are defined by Krall and Frink **(3)** by the equation

$$(1)\qquad \gamma_n(x, a, b) = {}_2F_0\left(-n, a + n - 1; -\frac{x}{b}\right).$$

The purpose of this paper is to present some series of products of these polynomials when the two arguments are different as in the case of Legendre and Hermite polynomials. Such an explanation was given by Brafman **(2)**, namely:

$$(2)\qquad \sum_{r=0}^{n} \binom{n}{r} \frac{\Gamma(2 - a - n)}{\Gamma(2 - a - n - r)} \left(-\frac{x}{b} - \frac{y}{b}\right)^{2r} \gamma_r\left(\frac{xy}{x + y}, a, b\right) = \gamma_n(x, a, b)\, \gamma_n(y, a, b).$$

These series will be stated and proved in § 2. The following formulae are required in the proofs:

$$(3)\qquad {}_4F_3\left(\begin{matrix}\alpha, 1 + \frac{1}{2}\alpha, d, e\\ \frac{1}{2}\alpha, 1 + \alpha - d, 1 + \alpha - e\end{matrix}; -1\right) = \frac{\Gamma(1 + \alpha - d)\, \Gamma(1 + \alpha - e)}{\Gamma(1 + \alpha)\, \Gamma(1 + \alpha - d - e)}$$

(1, p. 28, formula 3**)**. Gauss's theorem, namely:

If $R(\gamma) > 0$, $R(\gamma - \alpha - \beta) > 0$,

$$(4)\qquad F(\alpha, \beta; \gamma; 1) = \frac{\Gamma(\gamma)\, \Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\, \Gamma(\gamma - \beta)}$$

(4, p. 144, Example 2**)** and Saalschutz's theorem **(5)**, namely:

If $\rho + \sigma = \alpha + \beta + \gamma + 1$ and if α, β, or γ is a negative integer,

$$(5)\qquad F\left(\begin{matrix}\alpha, \beta, \gamma;\\ \rho, \sigma\end{matrix}\ 1\right) = \frac{\Gamma(\rho)\, \Gamma(1 + \alpha - \sigma)\, \Gamma(1 + \beta - \sigma)\, \Gamma(1 + \gamma - \sigma)}{\Gamma(1 - \sigma)\, \Gamma(\rho - \alpha)\, \Gamma(\rho - \beta)\, \Gamma(\rho - \gamma)}.$$

Frequent use will also be made of the factorial notation:

$$(\alpha; n) = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)} = \alpha(\alpha + 1) \ldots (\alpha + n - 1) \text{ for } n = 1, 2, 3, \ldots,$$

$$(\alpha, 0) = 1 \qquad\qquad \text{for } \alpha \neq 0.$$

2. Formulae and proofs: The expansions to be proved are:

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(6) $$\sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{\Gamma(1-a-n-r)}{\Gamma(2-a-r)} (1-a-2r)\, \gamma_r(x,a,b)\, \gamma_r(y,a,b) = \left(\frac{x}{b}+\frac{y}{b}\right)^n \gamma_n\left(\frac{xy}{x+y}, a, b\right);$$

(7) $$\sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{\Gamma(2-a-n-r)}{\Gamma(4-2a-2n-2r)} \left(\frac{x}{b}\right)^r \gamma_r(x,a,b) = \frac{\Gamma(2-a-n)}{\Gamma(4-a-2n)} \gamma_{2n}(x, 2a-2, b);$$

(8) $$\sum_{r=0}^{n} \binom{n}{r} \frac{\Gamma(2-a)}{\Gamma(2-a-r)} \left(-\frac{2x}{b}\right)^r \gamma_r(\tfrac{1}{2}x, a, b) = \gamma_{2n}(x, a-2n, b);$$

(9) $$\sum_{r=0}^{n} (-1)^r \frac{(1-a-2n+2r)\,\Gamma(1-a-2n+r)}{r!\{(n-r)!\,\Gamma(2-a-n+r)\}^2} \{\gamma_{n-r}(x,a,b)\}^2 = \{(n!)^2\,\Gamma(2-a)\}^{-1} \gamma_{2n}(x, a-2n, b);$$

(10) $$\sum_{r=0}^{n} \binom{n}{r} \frac{(1+\alpha;r)(1-a-\alpha-n-r;r)(1-a-2r)}{(1-a-n-r;n+1)} \gamma_r(x,a,b). = \gamma_n(x, 1+a+\alpha, b).$$

Proof of (6). If on the left of (6) we replace $\gamma_r(x,a,b)\,\gamma_r(y,a,b)$ by the series in (3), then after changing the order of summation it becomes

$$\sum_{m=0}^{n} (-1)^m \left(\frac{x}{b}+\frac{y}{b}\right)^m \gamma_m\left(\frac{xy}{x+y}, a, b\right) \sum_{r=m}^{n} (-1)^r \binom{n}{r}\binom{r}{m} \frac{\Gamma(1-a-n-r)}{\Gamma(2-a-m-r)} (1-a-2r).$$

If we apply the formula

$$\sum_{r=m}^{n} (-1)^r \binom{n}{r}\binom{r}{m} \frac{\Gamma(1-a-n-r)}{\Gamma(2-a-m-r)} (1-a-2r) = (-1)^n \delta_{mn},$$

then the right-hand side of (6) is obtained. If $y = x$, (6) becomes

(11) $$\sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{\Gamma(1-a-n-r)}{\Gamma(2-a-r)} (1-a-2r) \{\gamma_r(x,a,b)\}^2 = \left(\frac{2x}{b}\right)^n \gamma_n(\tfrac{1}{2}x, a, b).$$

Proof of (7). The following lemma is required in the proofs of (7) and (8):

LEMMA: *If n, N, r are positive integers, then*

(12) $$(-n; N-n+2r) = (-1)^n\{(2n)!\}^{-1}(n!)2^{2r}(-2n;N)\,(\tfrac{1}{2}N-n;r)\,(\tfrac{1}{2}+\tfrac{1}{2}N-n;r).$$

Proof. We have

$$(-n; N-n+2r) = 2^{N-n+2r}(-\tfrac{1}{2}n; \tfrac{1}{2}N-\tfrac{1}{2}n+r).$$

But

$$(-\tfrac{1}{2}n;\tfrac{1}{2}N-\tfrac{1}{2}n+r)=(-\tfrac{1}{2}n;\tfrac{1}{2}N-\tfrac{1}{2}n)(\tfrac{1}{2}N-n;r)$$

and

$$(\tfrac{1}{2}-\tfrac{1}{2}n;\tfrac{1}{2}N-\tfrac{1}{2}n+r)=(\tfrac{1}{2}-\tfrac{1}{2}n,\tfrac{1}{2}N-\tfrac{1}{2}n)\,(\tfrac{1}{2}+\tfrac{1}{2}N-n;r),$$

so that

$$\begin{aligned}&\frac{2^{n-N-2r}(-n;N-n+2r)}{(\frac{1}{2}N-n;r)(\frac{1}{2}+\frac{1}{2}N-n;r)}\\ &=(-\tfrac{1}{2}n;\tfrac{1}{2}N-\tfrac{1}{2}n)(\tfrac{1}{2}-\tfrac{1}{2}n;\tfrac{1}{2}N-\tfrac{1}{2}n)\\ &=2^{n-N}(-n;N-n)\\ &=\frac{\{(2n)(2n-1)\dots(n+1)\}\{(-n;N-n)\}\{(n!)\}}{2^{N-n}(2n)!}\\ &=\frac{(-1)^n n!}{2^{N-n}(2n)!}\{(-2n)(-2n+1)\dots(-2n+N-1)\}\\ &=(-1)^n 2^{n-N}(n!)\,\{(2n)!\}^{-1}\,\{(-2n;N)\},\end{aligned}$$

from which the lemma follows.

Proof of (7). In the left-hand side of (7) write $n-r$ for r, substitute for $\gamma_{n-r}(x,a,b)$ from (1), then it becomes:

$$\sum_{r=0}^{n}\sum_{s=0}^{n-r}(-1)^{n-r+s}\frac{\Gamma(2-a-2n+r)(-n+r;s)}{r!\,s!\,(n-r)!}\,\frac{(a+n-r-1;s)}{\Gamma(4-2a-4n+2r)}\left(\frac{b}{x}\right)^{r-s}.$$

Here we put $s=r+N-n$, and the last expression becomes:

$$\begin{aligned}&\frac{1}{n!\,\Gamma(a+n-1)}\sum_{N=0}^{2n}\sum_{r=0}^{[n-\frac{1}{2}N]}(-1)^N\frac{(-n;N-n+2r)\,\Gamma(2-a-2n+r)}{r!\,\Gamma(4-2a-4n+2r)}\\ &\qquad\frac{\Gamma(a+N-1)(2-a-n;r)}{\Gamma(1+N-n+r)}\left(\frac{b}{x}\right)^{n-N}\\ &=(-1)^n\frac{2^{3n+2a-2}\,\pi^{\frac{1}{2}}}{(2n)!\,\Gamma(\frac{5}{2}-a-2n)\,\Gamma(a+n-1)}\sum_{N=0}^{2n}(-2)^N\\ &\qquad\Gamma(a+N-1)(-2n;N)\left(\frac{b}{x}\right)^{n-N}\{\Gamma(1+N-n)\}^{-1}\\ &\qquad\times{}_3F_2\left[\begin{matrix}\frac{1}{2}N-n,\frac{1}{2}+\frac{1}{2}N-n,2-a-n\\1+N-n,\frac{5}{2}-a-2n\end{matrix};1\right]\end{aligned}$$

by Lemma (12).

Proof of (8). In the left-hand side of (8), write $n-r$ for r, substitute for $\gamma_{n-r}(\frac{1}{2}x,a,b)$ from (1), then it becomes

$$\sum_{r=0}^{n}\sum_{s=0}^{n-r}(-2)^{n-r-s}\binom{n}{r}\frac{\Gamma(2-a)(-n+r;s)(a+n-r-1;s)}{s!\,\Gamma(2-a-n+r)}\left(\frac{b}{x}\right)^{r-s-n}.$$

Here we put $s = r + N - n$; then the last expression becomes

$$\frac{2^{2n}\,\Gamma(2-a)}{\Gamma(a+n-1)\,\Gamma(2-a-n)} \sum_{N=0}^{2n} (-2)^{-N} \frac{\Gamma(a+N-1)}{\Gamma(1+N-n)} (-2n; N) \left(\frac{b}{x}\right)^{-N}$$
$$\times {}_2F_1\left[\begin{matrix} \tfrac{1}{2}N - n, \tfrac{1}{2} + \tfrac{1}{2}N - n; 1 \\ 1 + N - n \end{matrix}\right]$$

by Lemma (12). Here we sum the ${}_2F_1$ by Gauss's theorem (4) and so obtain (8). In (8) write $2x$ for x; then it becomes

$$(13)\quad \gamma_{2n}(2x, a-2n, b) = \sum_{r=0}^{n} \binom{n}{r} \frac{\Gamma(2-a)}{\Gamma(2-a-r)} \left(\frac{-4x}{b}\right)^r \gamma_r(x, a, b)$$

which may be taken as the duplication formula for Bessel polynomials.

Proof of (9). If $y = x$ in (2), then it becomes

$$(14)\quad \sum_{r=0}^{n} \binom{n}{r} \frac{\Gamma(2-a-n)}{\Gamma(2-a-n-r)} \left(\frac{-2x}{b}\right)^r \gamma_r(\tfrac{1}{2}x, a, b) = \{\gamma_n(x, a, b)\}^2.$$

To prove (9), substitute in the left of it for $\{\gamma_{n-r}(x, a, b)\}^2$ by the series in (14), then after some rearrangement the left-hand side of (9) becomes

$$\sum_{r=0}^{n} (-1)^r \frac{\Gamma(2-a-2n+2r)\,\Gamma(1-a-2n+r)}{r!(n-r)!\,\Gamma(1-a-2n+2r)\,\Gamma(2-a-n+r)} \left(\frac{b}{x}\right)^{-2n+2r}$$
$$\times \sum_{s=0}^{n-r} \frac{1}{s!(r-s)!\,\Gamma(2-a-2n+2r+s)} \left(\frac{b}{x}\right)^{2s} \gamma_{n-r-s}(\tfrac{1}{2}x, a, b)$$
$$= \frac{1}{n!}\left(\frac{b}{x}\right)^{-n} \sum_{p=0}^{n} (-2)^{n-p} \frac{\Gamma(2-a-2n)}{p!(n-p)!\,\Gamma(2-a-2n+p)\,\Gamma(2-a-n)}$$
$$\times \left(\frac{b}{x}\right)^p \gamma_{n-p}(\tfrac{1}{2}x, a, b)\, {}_4F_3\left[\begin{matrix} 1+a-2n, \tfrac{3}{2} - \tfrac{1}{2}a - n, -p, -n \\ \tfrac{1}{2} - \tfrac{1}{2}a - n, 2-a-n, 2-a-2n+p \end{matrix}; 1\right].$$

But the ${}_4F_3$ can be summed by (3) if we substitute in (3) $\alpha = 1 - a - 2n$, $d = -n$, $e = -p$. Thus the last expression reduces to

$$\frac{1}{n!}\left(\frac{b}{x}\right)^{-n} \sum_{p=0}^{n} \frac{(-2)^{n-p}}{p!(n-p)!\,\Gamma(2-a-n+p)} \left(\frac{b}{x}\right)^p \gamma_{n-p}(\tfrac{1}{2}x, a, b)$$
$$= \{(n!)^2\,\Gamma(2-a)\}^{-1} \sum_{r=0}^{n} \binom{n}{r} \frac{\Gamma(2-a)}{\Gamma(2-a-r)} \left(\frac{-2x}{b}\right)^r \gamma_r(\tfrac{1}{2}x, a, b)$$
$$= \{(n!)^2\,\Gamma(2-a)\}^{-1} \gamma_{2n}(x, a-2n, b)$$

by (8). Hence the proof of (9) is complete.

Proof of (10). To prove (10), substitute for $\gamma_r(x, a, b)$ in the left-hand side of (10) from (1), change the order of summation, then sum the innermost series by means of (3) and so obtain the right-hand side of (10) by a second application of (1).

Finally it may be noted that each of the explicitly summed series in the last proofs can be transformed to another explicitly summed series and the result in the two cases is the same.

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